# A generalization of Castigliano's theorems for structures with eigenstrains*

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**Abstract**

We prove two results that generalize the two classical theorems of Castigliano to structures with known (stationary) eigenstrains, including those originating from temperature changes, lack-of-fit, phase changes, etc. We show that, in these situations, the classical theorems of Castigliano still hold, provided that the appropriate modifications are made to the elastic energy and its conjugate. The derivation of these results follows directly from the principle of minimum potential energy and some of the properties of convex functions. Just like the classical theorems, the final results can be conveniently employed to solve simple structures by hand.

## 1 Introduction

The two theorems of Castigliano are popular results employed for solving simple problems in structural analysis and strength of materials (see, for example, [20, 18, 11]). In standard texts for these disciplines, the theorems are almost invariably studied for isothermal structures under the condition that they are free of stresses when undeformed. These restrictions are unfortunate because they disregard situations that happen frequently: structures under thermal loads, those possessing members with lack-of-fit, or deformed due to water absorption, etc. In these situations, solutions based on Castigliano's theorems must be abandoned for other, more laborious ones.

Already in Castigliano's original work, the two theorems that bear his name were presented only for isothermal structures without initial strains. Since then, the theorems that bear Castigliano's name have mainly remained restricted to

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these specific situations, motivating the analysis presented in the current article. It is our goal to show that extended versions of these two theorems can be proven in a quite straightforward way: these new statements will encompass the analysis of structures under *known*, although not necessarily uniform, eigenstrains, reducing to the original Castigliano theorems when these effects vanish.

Energy plays a crucial role in the statement and proof of Castigliano's theorems. In his dissertation [7], Castigliano presented for the fist time what today is known as his second theorem and referred to the *molecular work* as being minimized by the equilibrium solution. This concept, which we would call today the *internal* or *elastic energy*, was later employed by Castigliano himself to state and prove two theorems, showing that the applied forces are equal to the derivatives of the elastic energy with respect to the effective displacements, and viceversa [8] (see [1] for an English translation of the precise statements by Castigliano). With time, it has become clear that the second theorem actually involves the derivative of the *complementary elastic energy*, that turns out to be identical to the internal energy for linearly elastic structures (see, among many, [18, 20, 11]).

Extensions of Castigliano's second theorem for heated bodies have been investigated in the past. In [13, 17], a generalized version of Castigliano's second theorem for continua is proposed by appending the (isothermal) internal energy with a term resulting from the multiplication of the trace of the stress tensor and the thermal dilatation. This functional is shown to be the one whose variations with respect to the applied loading yield the conjugate displacement, thus acting as the fundamental potential for the extended version of Castigliano's theorem. In [2], extensions of Castigliano's theorems are derived for nonlinear bodies with thermal effects, again for continua, using similar arguments as the previous refernences. In [12] the authors arrive to the same extension using the generalized reciprocity theorem of Maysel [16]. In a related work, Biot obtained a generalized version of Castigliano's second theorem for strongly coupled thermoelastic materials under imposed temperature increments at discrete points [6]. This result is not applicable to the problems we have in mind, where possibly all points of the structure are subjected to inhomogeneous thermal jumps and strains due to lack-of-fit. However, it was later proved [3], that in the absence of heat sources and sinks, Biot's results reduced to those previously obtained by Hemp [13]. In the monograph by Ziegler [21], the theorems of Castigliano are studied for structures in the isothermal and thermal cases, much in the spirit of the current work. Following a different avenue, and not including other possible eigenstrains, the results obtained by Ziegler coincide with the ones proved herein. In a similar vein, Kovalenko's book on thermoelasticity [15] also contains a theorem for *continuum* thermoelasticity which can be associated with Castigliano's second theorem, and derived directly from equilibrium considerations. Similarly, Salençon [19] employed the principle of virtual displacements and the convexity of the complementary stored energy function to obtain a generalization of Castigliano's second theorem for thermally loaded structures leading to a result that is essentially identical to Ziegler's and to the one derive later in the article.

In contrast with the works mentioned above, in the current article we present and prove extensions of *both* theorems of Castigliano for structures under any type of eigenstrains, including thermal, lack-of-fit, and phase change strains. In fact, we stress that the nature of these eigenstrains is irrelevant, as long as they are known and are not the result of any force. Moreover, the results are derived and proved directly to beam structures, where they are most natural and useful and not for continua, where the concept of concentrated forces and torques is, at least, elusive. More importantly, we identify the precise energies involved in the two theorems, and recognize their relation to the classical theorems of Castigliano. The final results provide *explicit* expressions for the energies involved in Castigliano's theorems, energies that can be employed in a direct manner to accommodate all types of eigenstrains in structures.

As we will show, the the extended theorems of Castigliano follow in a straightforward manner from the principle of minimum potential energy and the properties of convex functions. In this respect, we will stress the role played by convex duality in the proof of the theorems, a route that will also help to give meaning to the energies involved in them and was not exploited by previous works on the subject.

The main result of the article, summarized in the form of a theorem, is very simple to apply to practical examples, and very general. For all bar and beam models, the proposed theorem encompasses all types of eigenstrains simply by calculating the correct value of the elastic and complementary elastic energies. In all these cases, the analytical expression of these energies is well known and easy to calculate.

The rest of the article is organized as follows. In section 2 the two classical theorems of Castigliano are presented and proved. This allows to set the notation, first, and to suggest a straightforward proof for the second theorem which will be later extended. The extension of the theorems is addressed in section 3, and their application to a simple example with beams in section 4. Section 5 closes the article with a discussion on the uses of Castigliano's theorems.

## 2 The classical theorems of Castigliano

We briefly present and proof the two classical theorems of Castigliano for elastic structures. The presentation is not the shortest one possible, but introduces the concepts that will most easily allow for an extension to general problems including thermal and pre-stress effects.

### 2.1 The principle of minimum potential for elastic structures

We consider in this article only structures of elastic materials, under the hypothesis of small strains, and consisting of one-dimensional members (beams/bars). In such models, the fundamental entity where equilibrium and deformation are defined is that of the cross section and we will use the symbol $s$ to denote a scalar that uniquely identifies each of the sections within a structure.

The kinematics of the deformable structures considered herein is completely described by the generalized displacement of its cross section $\boldsymbol{u} = \boldsymbol{u}(s)$. From this field, a vector field of strain measures $\boldsymbol{E}$ can be determined and we write $\boldsymbol{E}(s) = \bar{\boldsymbol{E}}(\boldsymbol{u}(s))$ for the linear displacement-strain relation. The specific definition of the strain measures depends on the structural theory under consideration for each member (two- vs. three-dimensional model, Euler-Bernoulli vs. Timoshenko beam theory, etc.). At this stage of the presentation, the precise choice of strain measures is not relevant, but we assume that they are work-conjugate to stress resultants, collected in a vector denoted generically as $\boldsymbol{\Sigma}$, and that there always exists a linear constitutive relation of the form

$$\boldsymbol{\Sigma} = \boldsymbol{C}(s)\boldsymbol{E} \ , \tag{1}$$

The matrix $\boldsymbol{C}$, depends, in general, for each cross section of the structure, of its geometry and the properties of the material at its points. The elastic energy per unit length can be thus defined as the quadratic function

$$\psi(\boldsymbol{E}, s) = \frac{1}{2}\boldsymbol{E} \cdot \boldsymbol{C}(s)\boldsymbol{E}, \tag{2}$$

and the constitutive relation (1) might be rewritten as

$$\boldsymbol{\Sigma} = \frac{\partial \psi(\boldsymbol{E}, s)}{\partial \boldsymbol{E}} \ . \tag{3}$$

Consider an elastic structure under a distributed load field $\boldsymbol{f} = \boldsymbol{f}(s)$. Upon the application of loading, the structure is deformed and the work done by the loads is stored in the form of *elastic energy* $V_{int}$, which is defined as a function

$$V_{int}(\boldsymbol{u}) = \int \psi(\bar{\boldsymbol{E}}(\boldsymbol{u}(s)), s) \, \mathrm{d}s \ . \tag{4}$$

In this equation, and below, the integral is performed over all structural members. Similarly, the external energy of the applied loads is defined as

$$V_{ext}(\boldsymbol{u}) = \int \boldsymbol{f}(s) \cdot \boldsymbol{u}(s) \, \mathrm{d}s \ . \tag{5}$$

The *principle of minimum potential energy* states that the total potential energy of the system

$$V(\boldsymbol{u}) = V_{int}(\boldsymbol{u}) - V_{ext}(\boldsymbol{u}) \tag{6}$$

is a minimum at the equilibrium displacements. The convexity of the elastic energy density guarantees that there exists a minimizer of the potential energy and the optimality condition for the displacements is

$$\boldsymbol{0} = \frac{\delta V}{\delta \boldsymbol{u}}(\boldsymbol{u}) \ , \tag{7}$$

where the operation ${}^{\delta}/_{\delta \boldsymbol{u}}$ refers to the variational derivative.

## 2.2 Structures under point loads

Consider an elastic structure as before under a finite number of point forces and concentrated moments, denoted generically as $\boldsymbol{f}_i$, applied on the cross sections $s_i$, for $i = 1, 2, \ldots, N$. The corresponding conjugate (generalized) displacements will be denoted $\boldsymbol{u}_i$. When $\boldsymbol{f}_i$ is a point force, $\boldsymbol{u}_i$ corresponds to a displacement; if $\boldsymbol{f}_i$ is a concentrated moment, $\boldsymbol{u}_i$ refers to a rotation vector.

Let $\boldsymbol{\nu}_i$ be the unit vector in the direction of $\boldsymbol{f}_i$. The (generalized) *effective displacement* $\Delta_i$ is the projection of $\boldsymbol{u}_i$ onto $\boldsymbol{\nu}_i$, i.e.,

$$\Delta_i = \boldsymbol{u}_i \cdot \boldsymbol{\nu}_i = \boldsymbol{u}_i \cdot \frac{\boldsymbol{F}_i}{|\boldsymbol{F}_i|} = \boldsymbol{u}_i \cdot \frac{\boldsymbol{F}_i}{F_i} \tag{8}$$

where we have used the notation $F_i = |\boldsymbol{F}_i|$. For later use, let us collect all the generalized forces moduli and effective displacements in two vectors of dimension $N$, denoted respectively as $\boldsymbol{\Phi}$ and $\boldsymbol{\Delta}$ with components $(\boldsymbol{\Phi})_i = F_i$ and $(\boldsymbol{\Delta})_i = \Delta_i$.

The displacement field $\boldsymbol{u}$ and strain measures $\boldsymbol{E}$ at each cross section of the structure can be uniquely determined as a function of the loading, geometry, and material properties. Assuming that the directions of application of each force, denoted previously as $\boldsymbol{\nu}_i$, is fixed, and since the geometry and material properties are constant, there exist functions $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{E}}$, and $\tilde{\boldsymbol{\Sigma}}$ such that

$$\boldsymbol{u}(s) = \tilde{\boldsymbol{u}}(s, \boldsymbol{\Phi}) \ , \qquad \boldsymbol{E}(s) = \tilde{\boldsymbol{E}}(s, \boldsymbol{\Phi}), \qquad \boldsymbol{\Sigma}(s) = \tilde{\boldsymbol{\Sigma}}(s, \boldsymbol{\Phi}). \tag{9}$$

In view of the linearity of the structure, the three functions must be linear in their second argument. Moreover, the effective displacements can be obtained simply by projecting $\boldsymbol{u}(s_i)$ onto the direction $\boldsymbol{\nu}_i$ and thus

$$\Delta_i = \tilde{\boldsymbol{u}}(s_i, \boldsymbol{\Phi}) \cdot \boldsymbol{\nu}_i. \tag{10}$$

Due to the linearity of the function $\tilde{\boldsymbol{u}}$ on its second argument, we deduce that there exists a *flexibility matrix* $\boldsymbol{S}$ that relates force moduli and effective displacements. Defining its inverse, the *stiffness matrix* $\boldsymbol{K}$, we can write

$$\boldsymbol{\Delta} = \boldsymbol{S}\boldsymbol{\Phi}, \qquad \boldsymbol{\Phi} = \boldsymbol{K}\boldsymbol{\Delta}. \tag{11}$$

In view of the previous relation, we can find expressions for the displacement and strain measures, at each point of the structure, that are functions of the effective displacements, that is,

$$\boldsymbol{u}(s) = \hat{\boldsymbol{u}}(s, \boldsymbol{\Delta}) = \tilde{\boldsymbol{u}}(s, \boldsymbol{K}\boldsymbol{\Delta}), \qquad \boldsymbol{E}(s) = \hat{\boldsymbol{E}}(s, \boldsymbol{\Delta}) = \tilde{\boldsymbol{E}}(s, \boldsymbol{K}\boldsymbol{\Delta}). \tag{12}$$

Finally, making explicit the linearity of the function $\hat{\boldsymbol{E}}$ with respect to its second argument, we write

$$\boldsymbol{E}(s) = \boldsymbol{B}(s)\boldsymbol{\Delta}, \tag{13}$$

a relation that will play an important role in the derivation of Castigliano's theorems.

The previous derivation motivates the following remarks:

*i*) The flexibility and stiffness matrices defined previously are not the ones typically employed in structural analysis that express the relations between the $M$ nodal displacements $\{\boldsymbol{u}_i\}_{i=1}^M$ and the conjugate nodal forces $\{\boldsymbol{f}_i\}_{i=1}^M$. The matrices $\boldsymbol{S}, \boldsymbol{K}$ defined above have dimensions equal to the number of applied forces, and relate scalar quantities $F_i$ and $\Delta_i$.

*ii*) We have used the fact that the flexibility matrix is invertible, a fact that follows from the stability of the structure: from Clapeyron's theorem, the work performed on the structure by all the loads, denoted $W_{ext}$ and with the same value as the elastic energy, is

$$W_{ext} = \frac{1}{2}\boldsymbol{\Phi}\cdot\boldsymbol{\Delta} = \frac{1}{2}\boldsymbol{\Phi}\cdot\boldsymbol{S}\boldsymbol{\Phi} = \frac{1}{2}\boldsymbol{\Delta}\cdot\boldsymbol{K}\boldsymbol{\Delta} = V_{int}. \tag{14}$$

If this quantity is non-negative for every force and displacement vectors, the flexibility and stiffness matrices need to be positive-definite, and hence invertible.

*iii*) The matrix $\boldsymbol{B}$ relates the effective displacements with the strains at every point of the structure. This is not to be mistaken with the strain-displacement matrix employed in the stiffness method of structural analysis or the finite element method. In the later case, the matrix operates on the displacements of all the nodes of the structure; in the former, in contrast, the matrix operates only on the effective displacements defined only at the points where the loads are applied.

*iv*) The strain-displacement matrix of structural and finite element analysis can be constructed using only geometrical arguments. In contrast, the matrix $\boldsymbol{B}$ defined in Eq. (13) involves information from the material, equilibrium, etc. As deduced from its definition, the $\boldsymbol{B}$ matrix requires, for its calculation, the operator $\hat{\boldsymbol{E}}$, which can only be obtained after the flexiblity coefficients are known.

## 2.3 The classical theorems of Castigliano

We present and proof next the two classical theorems of Castigliano in a way that is most useful for later extensions. There are simpler proofs than the one presented herein (for example in [18]), but we have chosen the arguments that highlight the role of the strain matrix $\boldsymbol{B}$ and the strain-effective displacement relations. This avenue paves the way to introducing the effects of thermal and pre-strains in the application of the theorem.

When the structure is elastic and it is under concentrated loads $\{\boldsymbol{f}_i\}_{i=1}^N = \{F_i\,\boldsymbol{\nu}_i\}_{i=1}^N$, the results of the previous section allows define the potential energy

$$\hat{V}(\boldsymbol{\Delta}) = V(\hat{\boldsymbol{u}}(\boldsymbol{\Delta})) = V_{int}(\hat{\boldsymbol{u}}(\boldsymbol{\Delta})) - \boldsymbol{\Phi}\cdot\boldsymbol{\Delta} = \hat{V}_{int}(\boldsymbol{\Delta}) - \boldsymbol{\Phi}\cdot\boldsymbol{\Delta}\ . \tag{15}$$

Then, the derivative of the potential energy $\hat{V}$ with respect to the vector of effective displacements gives, by the chain rule,

$$\frac{\partial \hat{V}(\boldsymbol{\Delta})}{\partial \boldsymbol{\Delta}} = \left(\frac{\partial \hat{\boldsymbol{u}}(\boldsymbol{\Delta})}{\partial \Delta}\right)^T \frac{\delta V(\boldsymbol{u}(\boldsymbol{\Delta}))}{\delta \boldsymbol{u}} \ , \tag{16}$$

which must be zero, due to Eq. (7). Splitting the potential energy into its two parts we obtain

$$\mathbf{0} = \frac{\partial \hat{V}(\boldsymbol{\Delta})}{\partial \boldsymbol{\Delta}} = \frac{\partial \hat{V}_{int}(\boldsymbol{\Delta})}{\partial \boldsymbol{\Delta}} - \boldsymbol{\Phi}. \tag{17}$$

Hence, we conclude

$$\boldsymbol{\Phi} = \frac{\partial \hat{V}_{int}(\boldsymbol{\Delta})}{\partial \boldsymbol{\Delta}}, \tag{18}$$

which is the statement of Castigliano's first theorem.

Due to the hypotheses of the model, the internal energy function $\hat{V}_{int}$ can be evaluated as follows:

$$\begin{aligned} \hat{V}_{int}(\boldsymbol{\Delta}) &= \int \frac{1}{2} \hat{\boldsymbol{E}}(s, \boldsymbol{\Delta}) \cdot \boldsymbol{C}(s) \hat{\boldsymbol{E}}(s, \boldsymbol{\Delta}) \, \mathrm{d}s \\ &= \frac{1}{2} \boldsymbol{\Delta} \cdot \int \boldsymbol{B}^T(s) \boldsymbol{C}(s) \boldsymbol{B}(s) \, \mathrm{d}s \ \boldsymbol{\Delta}. \end{aligned} \tag{19}$$

The function $\hat{V}_{int}$ is quadratic and, moreover, using Eqs. (11) and (14) we conclude that the integral in Eq. (19) corresponds to the stiffness matrix $\boldsymbol{K}$.

More importantly, since $\hat{V}_{int}$ is quadratic, it is convex. Its convex conjugate $\hat{V}^*_{int} = \hat{V}^*_{int}(\boldsymbol{\Phi})$, is the complementary elastic energy. This function satisfies, by duality,

$$\boldsymbol{\Delta} = \frac{\partial \hat{V}^*_{int}(\boldsymbol{\Phi})}{\partial \boldsymbol{\Phi}}, \tag{20}$$

which is the statement of Castigliano's second theorem.

This simple proof of Castigliano's theorems highlights the key role played by the convexity of the elastic energy, and duality. These same concepts will allow a straightforward generalization of these classical results to situations in which the structure might go through thermal deformations and lack-of-fit.

We remark that, in textbooks, it is often stated that Eq. (17) is the statement of the principle of mininum potential energy. In fact, this relation is a corollary of the true result (7). For a given structure, the optimality condition (7) gives the displacement at equilibrium of all points. The condition (17), in contrast, gives only partial information about the deformed state of the structure and can not be considered to be a fundamental variational principle that replaces the equations of equilibrium, like the principle of minimum potential energy is known to be.

# 3 Extensions of Castigliano theorems

Consider now a loaded structure whose members have some (known) eigenstrains, that is, strains not due to external forces. These might be caused by thermal increments, defects in their nominal dimensions, phase changes, water absorption, etc. The eigenstrain measures will be collectively denoted as $\boldsymbol{\theta}$ and might, in general, be different for each cross section. For deformable structures where the eigenstrains in the reference configuration are known, the fundamental variational principle is still the principle of minimum potential energy. In this case we write

$$V(\boldsymbol{u};\boldsymbol{\theta}) = V_{int}(\boldsymbol{u};\boldsymbol{\theta}) - V_{ext}(\boldsymbol{u}) \ , \tag{21}$$

where we have used the notation $V(\cdot;\boldsymbol{\theta}), V_{int}(\cdot;\boldsymbol{\theta})$ to denote that the eigenstrains $\boldsymbol{\theta}$ enter the two energies as fixed, known, fields.

The external potential energy is defined as in Eq. (5). The internal potential energy resembles the function defined in Eq. (4), but now accounts for the eigenstrains as follows:

$$V_{int}(\boldsymbol{u};\boldsymbol{\theta}) = \int \psi(\bar{\boldsymbol{E}}(\boldsymbol{u}(s)), \boldsymbol{\theta}(s), s)\,\mathrm{d}s. \tag{22}$$

The elastic energy density $\psi$ is defined, for all elastic structures, as

$$\psi(\boldsymbol{E},\boldsymbol{\theta},s) = \frac{1}{2}(\boldsymbol{E}-\boldsymbol{\theta})\cdot\boldsymbol{C}(s)(\boldsymbol{E}-\boldsymbol{\theta}) \ , \tag{23}$$

where, as in Section 2, the matrix $\boldsymbol{C}$ might be different for each cross section and we have taken, for convenience and without loss of generality, $\psi(\boldsymbol{0},\boldsymbol{\theta},s) = 0$. It bears emphasis that when the eigenstrains vanish, the elastic energy densities defined in Eqs. (2) and (23) coincide.

As in the situation described in Section 2, the displacement field $\boldsymbol{u}$ is the unknown of the problem, and since it is a minimizer of the potential energy, it must satisfy the optimality condition

$$\frac{\delta V(\boldsymbol{u};\boldsymbol{\theta})}{\delta \boldsymbol{u}} = \boldsymbol{0}. \tag{24}$$

For structures under point loads $\{\boldsymbol{f}_i\}_{i=1}^N$, the internal and external potential energies can be expressed as a function on the effective displacements. Following the reasoning of Section 2, we define the potential energy

$$\hat{V}(\boldsymbol{\Delta};\boldsymbol{\theta}) = V(\hat{\boldsymbol{u}}(\boldsymbol{\Delta});\boldsymbol{\theta}), \tag{25}$$

which is the difference of the two contributions

$$\hat{V}_{int}(\boldsymbol{\Delta};\boldsymbol{\theta}) = \int \psi(\hat{\boldsymbol{E}}(s,\boldsymbol{\Delta}),\boldsymbol{\theta})\,\mathrm{d}s, \qquad \hat{V}_{ext}(\boldsymbol{\Delta}) = \boldsymbol{\Phi}\cdot\boldsymbol{\Delta}. \tag{26}$$

The optimality condition (24) implies that, for structures under point loads

$$\boldsymbol{0} = \frac{\partial \hat{V}(\boldsymbol{\Delta};\boldsymbol{\theta})}{\partial \boldsymbol{\Delta}}. \tag{27}$$

Expressing the potential energy as the difference of the two terms in Eq. (26), the last result reads

$$\boldsymbol{\Phi} = \frac{\partial \hat{V}_{int}(\boldsymbol{\Delta};\boldsymbol{\theta})}{\partial \boldsymbol{\Delta}} \ . \tag{28}$$

This is the statement of Castigliano's first theorem in the presence of eigenstrains of arbitrary origin.

Using Eq. (13), we can write the internal energy showing the explicit dependence of the functional on the effective displacements, that is

$$\begin{aligned}\hat{V}_{int}(\boldsymbol{\Delta};\boldsymbol{\theta}) = \frac{1}{2}\int (\boldsymbol{B}(s)\boldsymbol{\Delta} - \boldsymbol{\theta}(s))\cdot \\ \boldsymbol{C}(s)(\boldsymbol{B}(s)\boldsymbol{\Delta} - \boldsymbol{\theta}(s))\,\mathrm{d}s.\end{aligned} \tag{29}$$

As in the case of the internal potential energy of section 2, the internal energy $\hat{V}_{int}$ is convex with respect to the effective displacements $\boldsymbol{\Delta}$. Thus, we can calculate its convex conjugate with respect to its first argument, that is, $\hat{V}^*_{int} = \hat{V}^*_{int}(\boldsymbol{\Phi};\boldsymbol{\theta})$ and by duality, Castigliano's theorem (as given by Eq. (28)) implies

$$\boldsymbol{\Delta} = \frac{\partial \hat{V}^*_{int}(\boldsymbol{\Phi};\boldsymbol{\theta})}{\partial \boldsymbol{\Phi}}, \tag{30}$$

which is precisely Castigliano's second theorem for structures with eigenstrains.

In elastic structures, the function $\hat{V}^*_{int}$ can be found explicitly, which makes the previous derivation not only theoretically appealing, but very convenient for calculating structures. The dual of $\hat{V}_{int}$ is obtained using the Legendre transform

$$\hat{V}^*_{int}(\boldsymbol{\Phi};\boldsymbol{\theta}) = \sup_{\boldsymbol{\Delta}} \left(\boldsymbol{\Phi}\cdot\boldsymbol{\Delta} - \hat{V}_{int}(\boldsymbol{\Delta};\boldsymbol{\theta})\right). \tag{31}$$

The optimal value of $\boldsymbol{\Delta}$ is obtained solving

$$\begin{aligned}\mathbf{0} &= \frac{\partial}{\partial \boldsymbol{\Delta}}\left(\boldsymbol{\Phi}\cdot\boldsymbol{\Delta} - \hat{V}_{int}(\boldsymbol{\Delta};\boldsymbol{\theta})\right) \\ &= \boldsymbol{\Phi} - \int \boldsymbol{B}(s)^T\boldsymbol{C}(s)\left(\boldsymbol{B}(s)\boldsymbol{\Delta} - \boldsymbol{\theta}(s)\right)\,\mathrm{d}s \\ &= \boldsymbol{\Phi} - \boldsymbol{K}\boldsymbol{\Delta} + \int \boldsymbol{B}(s)^T\boldsymbol{C}(s)\boldsymbol{\theta}(s)\,\mathrm{d}s,\end{aligned} \tag{32}$$

whose solution is

$$\boldsymbol{\Delta} = \boldsymbol{K}^{-1}\boldsymbol{\Phi} + \boldsymbol{K}^{-1}\int \boldsymbol{B}(s)^T\boldsymbol{C}(s)\boldsymbol{\theta}(s)\,\mathrm{d}s \tag{33}$$

Replacing this value for the optimal effective displacements in Eq. (31) we obtain, after some straightforward manipulations,

$$\begin{aligned}\hat{V}^*_{int}(\boldsymbol{\Phi};\boldsymbol{\theta}) &= \int \Big[\frac{1}{2}\tilde{\boldsymbol{\Sigma}}(s,\boldsymbol{\Phi})\cdot\boldsymbol{C}^{-1}(s)\tilde{\boldsymbol{\Sigma}}(s,\boldsymbol{\Phi}) + \\ &\qquad \tilde{\boldsymbol{\Sigma}}(s,\boldsymbol{\Phi})\cdot\boldsymbol{\theta}(s)\Big]\,\mathrm{d}s \\ &= \int \psi^*(\tilde{\boldsymbol{\Sigma}}(s,\boldsymbol{\Phi}),\boldsymbol{\theta})\,\mathrm{d}s,\end{aligned} \tag{34}$$

with $\psi^*$ the complementary internal energy density, given by

$$\psi^*(\boldsymbol{\Sigma}, \boldsymbol{\theta}) = \sup_{\boldsymbol{E}} (\boldsymbol{\Sigma} \cdot \boldsymbol{E} - \psi(\boldsymbol{E}, \boldsymbol{\theta})) = \frac{1}{2} \boldsymbol{\Sigma} \cdot \boldsymbol{C}^{-1} \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \cdot \boldsymbol{\theta} \ . \tag{35}$$

In structures free of eigenstrains (the ones considered in section 2), the internal potential energy $V_{int}$ and its dual $V^*_{int}$ have the same value at equilibrium, that is

$$\hat{V}_{int}(\boldsymbol{\Delta}) = \hat{V}^*_{int}(\boldsymbol{\Phi}) \tag{36}$$

when $\boldsymbol{\Phi} = \boldsymbol{K}\boldsymbol{\Delta}$. This is a consequence of the simple quadratic form of the internal energy. This coincidence makes the calculation of the complementary energy very simple and leads several textbooks to state that in the second theorem of Castigliano the effective displacements are obtained from the derivative of the internal energy *expressed in terms of the loads* (see, for example, [9, 4, 14, 10]). This fortunate coincidence does not happen for the general case studied in this section.

We summarize, for convenience, the main result of this section:

**Theorem 1** (Extended Castigliano's theorems)**.** *Let an elastic structure be subjected to (generalized) point loads* $\{\boldsymbol{f}\}_{i=1}^N$ *as well as eigenstrains* $\boldsymbol{\theta}$*. If* $\hat{V}_{int}$ *and* $\hat{V}^*_{int}$ *are the elastic and the complementary elastic energies, respectively, of the structure, then*

$$\boldsymbol{\Phi} = \frac{\partial \hat{V}_{int}(\boldsymbol{\Delta}; \boldsymbol{\theta})}{\partial \boldsymbol{\Delta}} \ , \qquad \boldsymbol{\Delta} = \frac{\partial \hat{V}^*_{int}(\boldsymbol{\Phi}; \boldsymbol{\theta})}{\partial \boldsymbol{\Phi}} \ , \tag{37}$$

*with* $\boldsymbol{\Phi}$ *being the vector of force moduli,* $\boldsymbol{\Delta}$ *the vector of (generalized) effective displacements. The energies* $\hat{V}_{int}$ *and* $\hat{V}^*_{int}$ *are defined respectively by*

$$\hat{V}_{int}(\boldsymbol{\Delta}; \boldsymbol{\theta}) = \int \psi(\hat{\boldsymbol{E}}(s, \boldsymbol{\Delta}), \boldsymbol{\theta}(s)) \, \mathrm{d}s, \tag{38a}$$

$$\hat{V}^*_{int}(\boldsymbol{\Phi}; \boldsymbol{\theta}) = \int \psi^*(\tilde{\boldsymbol{\Sigma}}(s, \boldsymbol{\Phi}), \boldsymbol{\theta}(s)) \, \mathrm{d}s, \tag{38b}$$

*with*

$$\psi(\boldsymbol{E}, \boldsymbol{\theta}, s) = \frac{1}{2} (\boldsymbol{E} - \boldsymbol{\theta}) \cdot \boldsymbol{C}(s) (\boldsymbol{E} - \boldsymbol{\theta}) \ , \tag{39a}$$

$$\psi^*(\boldsymbol{\Sigma}, \boldsymbol{\theta}, s) = \frac{1}{2} \boldsymbol{\Sigma} \cdot \boldsymbol{C}^{-1}(s) \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \cdot \boldsymbol{\theta} \ . \tag{39b}$$

# 4 Application to trusses and beams

We consider in this section a simple example that illustrates the extended version of Castigliano's second theorem, which is the most useful for the solution of simple problems in mechanics of solids.

Figure 1 depicts a structure consisting of a beam $ABC$ pinned to a bar $BD$. The beam section has inertia $I$, thickness $t$, and the bar has cross section

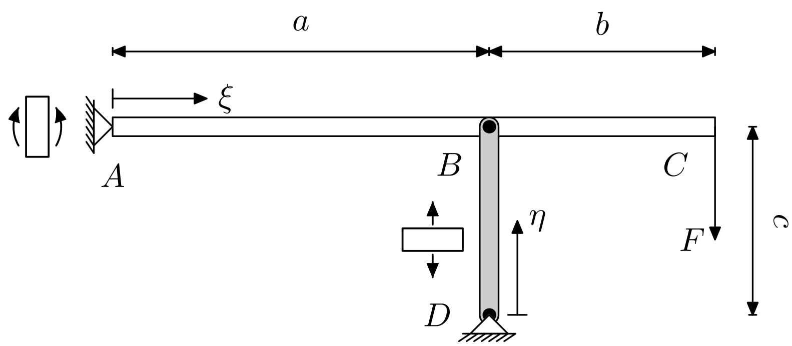


Figure 1: Example for section 4.

area $A$. Both members are of a material with Young's modulus $E$ and thermal expansion coefficient $\alpha$. The structure is pinned in $A$ and $D$, and subjected to a concentrated force $F$ on $C$. The geometric dimensions of the structure, as well as the positive sign for the bending moment and axial force are indicated in the figure.

Let us assume that the beam has been heated, and each cross section has a linear temperature profile with the temperatures at the bottom and the top being $\theta/2$ higher and lower, respectively, than the reference temperature. This temperature distribution creates a stress free curvature $\kappa_\theta = \alpha\theta/t$ that must be accounted for in addition to another stress free curvature, $\kappa_0$, due to defects in the initial shape of the beam as compared to its nominal, straight configuration. With all these ingredients, the bending moment $M$ in a cross section must verify:

$$M = EI(\kappa - \kappa_0 - \kappa_\theta) \ , \tag{40}$$

where $\kappa$ is the curvature of the beam and $\kappa_0 + \kappa_\theta$ is the bending eigenstrain.

Similarly, let us assume that the bar is heated causing a temperature increment $\bar{\theta}$ and thus an axial deformation $\varepsilon_\theta = \alpha\bar{\theta}$. If, additionally, the bar has a lack-of-fit longitudinal strain $\varepsilon_0$, the constitutive relation for the axial force $N$ in the bar is

$$N = EA(\varepsilon - \varepsilon_0 - \varepsilon_\theta) \ , \tag{41}$$

with $\varepsilon$ being the axial strain and $\varepsilon_0 + \varepsilon_\theta$ corresponding to the axial eigenstrain. We note that, following the notation of Section 3, the strain energy density of the beam and bar are, respectively,

$$\psi_{beam} = \frac{1}{2}EI(\kappa - \kappa_0 - \kappa_\theta)^2 \ , \quad \psi_{bar} = \frac{1}{2}EA(\varepsilon - \varepsilon_0 - \varepsilon_\theta)^2. \tag{42}$$

To calculate the deflection of the tip of the beam we use the extended version of Castigliano's second theorem. For that, we obtain first the expression for the bending moment in the beam and axial force in the bar. A straightforward analysis gives

$$\begin{aligned} M(\xi) &= -F\frac{b}{a}\langle\xi\rangle + F(1+\frac{b}{a})\langle\xi - a\rangle \\ N(\eta) &= -F(1+\frac{b}{a})\ , \end{aligned} \tag{43}$$

where $\langle\cdot\rangle$ is the Macaulay bracket and the coordinates $\xi, \eta$ are indicated in Figure 1. The complementary elastic energies per unit length in the beam and bar are, respectively,

$$\psi^*_{beam} = \frac{M^2}{2EI} + M(\kappa_\theta + \kappa_0)\ , \quad \psi^*_{bar} = \frac{N^2}{2EA} + N(\varepsilon_\theta + \varepsilon_0)\ . \tag{44}$$

Hence, the complementary internal energy of the complete structure is

$$\begin{aligned} \hat{V}^*_{int} = &\int_0^{a+b} \left(\frac{M^2}{2EI} + M(\kappa_\theta + \kappa_0)\right) \mathrm{d}\xi \\ &+ \int_0^c \left(\frac{N^2}{2EA} + N(\varepsilon_\theta + \varepsilon_0)\right) \mathrm{d}\eta. \end{aligned} \tag{45}$$

Finally, and according to Theorem 1, the effective displacement of the tip of the beam (that is, its downwards displacement) is

$$\begin{aligned} \Delta =& \frac{\partial \hat{V}^*_{int}}{\partial F} \\ =& \int_0^{a+b} \left(\frac{M}{EI} + \kappa_\theta + \kappa_0\right) \frac{\partial M}{\partial F}\,\mathrm{d}\xi \\ &+ \int_0^c \left(\frac{N}{EA} + \varepsilon_\theta + \varepsilon_0\right) \frac{\partial N}{\partial F}\,\mathrm{d}\eta \\ =& \frac{Fb^2(a+b)}{3EI} + \frac{F(a+b)^2c}{EA\,a^2} \\ &+ \int_0^{a+b} (\kappa_\theta + \kappa_0)\frac{\partial M}{\partial F}\,\mathrm{d}\xi + \int_0^c (\varepsilon_\theta + \varepsilon_0)\frac{\partial N}{\partial F}\,\mathrm{d}\eta \end{aligned} \tag{46}$$

If the thermal increments, both in the beam and the bar, are the same for all their cross sections, and the lack-of-fit strains are homogeneous as well, the previous expression evaluates to:

$$\begin{aligned} \Delta =& \frac{Fb^2(a+b)}{3EI} + \frac{F(a+b)^2c}{EA\,a^2} \\ &- \frac{(a+b)b}{2}(\kappa_\theta + \kappa_0) - \frac{(a+b)c}{a}(\varepsilon_\theta + \varepsilon_0). \end{aligned} \tag{47}$$

This result can be verified using standard arguments based on kinematics and equilibrium.

Let us close this section by noting that the exactly the same procedure can be employed to account for thermal or other types of eigenstrains in torsion, out-of-plane bending, etc. Since all the elastic energies for bars and beams are quadratic and of the form (39a), accounting for the stress-free strains is as simple, in every case, as it is done in Eq. (44) for the example showed in this section.

# 5 Discussion

In his 1873 dissertation [7], Alberto Castigliano stated, for the first time, what today is referred to as his second theorem. In a later work [8], he presented and proved the two theorems that currently bear his name. In a English translation of the latter work published in the early 1900's [1], Castigliano's text was presented in the following way:

> *"Theorem of the Differential Coefficients of the Internal Work. Part 1 – If the internal work of a framed structure is expressed as a function of the relative displacements of the external forces applied to its nodes, the resulting expression is such that its differential coefficients with regard to these displacements give the values of the corresponding forces. Part 2 – If, on the contrary, the internal work of a framed structure is expressed as a function of the external forces, the resulting expression is such that its differential coefficients give the relative displacements of their points of application."*

Many books present Castigliano's theorems using similar wordings to this one. See, for example, [9, 4, 14, 10]. This kind of statements is correct for problems without eigenstrains, but it misses one fundamental aspect. As we tried to explain in this article, the second theorem of Castigliano actually involves the complementary internal energy of the structure, not the internal energy. However, due to the simple quadratic form of the latter, the two energies have the same value at equilibrium. This point is stressed in many other texts (see, for example, [20, 5, 11]).

Structures that have eigenstrains still verify two energy theorems, much in the spirit of the original results by Castigliano, as shown in Theorem 1. In these cases, however, the elastic energy is not as simple as in the situation alluded to before. Moreover, this energy and its complementary function fail to evaluate identically at equilibrium and thus a direct use of Castigliano's theorems in their classical wording will not be true. But even in this general case, the explicit form of the complementary internal energy can be found for any linear structure.

The most important practical result of the article, as summarized in Theorem 1, is that Castigliano theorems can still be used in structures with eigenstrains of arbitrary nature, as long as the correct elastic energy and its complementary function are employed. For beam and bars under all possible deformation modes, these two energies can be found in a simple closed form and explicitly gien in Eqs. (38) and (39)).

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