Mathematical preliminaries

Discretization Methods in Engineering

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Vector spaces

A (real) vector space $\mathcal{V}$ is a set together with two operations denoted '+' and multiplication by a scalar such that

- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
- $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
- $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
- For all $\boldsymbol{u} \in \mathcal{V}$, there exists a unique $(-\boldsymbol{u}) \in \mathcal{V}$, $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$
- $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}$, for all $\alpha \in \mathbb{R}$
- $(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}$
- $(\alpha\beta)\boldsymbol{u} = \alpha(beta\boldsymbol{u})$
- $1\boldsymbol{u} = \boldsymbol{u}$.

Norms and inner products

A norm on a (real) vector space is an operation $\|\cdot\| : \mathcal{V} \to \mathbb{R}$ such that

- $\|\boldsymbol{u}\| \geq 0$, $\|\boldsymbol{u}\| = 0 \leftrightarrow \boldsymbol{u} = \boldsymbol{0}$
- $\|\alpha \boldsymbol{u}\| = |\alpha| \, \|\boldsymbol{u}\|$
- $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$

A norm is the mathematical tool for measuring objects in vector spaces.
An (real) inner product on a (real) vector space is an operation $(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that

- $(\boldsymbol{v}, \boldsymbol{w}) = (\boldsymbol{w}, \boldsymbol{v})$,
- $(\alpha \boldsymbol{v} + \beta \boldsymbol{w}, \boldsymbol{u}) = \alpha(\boldsymbol{v}, \boldsymbol{u}) + \beta(\boldsymbol{w}, \boldsymbol{u})$,
- $(\boldsymbol{v}, \boldsymbol{v}) \geq 0, (\boldsymbol{v}, \boldsymbol{v}) = 0 \leftrightarrow \boldsymbol{v} = \boldsymbol{0}$.

An inner product defines a norm $\|\boldsymbol{u}\| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$.

Normed spaces / Banach spaces

A normed space is a pair $(\mathcal{V}, \|\cdot\|)$. This allows to define convergence of sequences: a sequence $\{x_n\} \in \mathcal{V}$ converges to x if

$$\lim_{n\to\infty} \|x - x_n\| = 0$$

It may happen that a sequence of elements that are closer and closer to each other does not converge. If sequences of this type always converge, the normed space is a Banach space.

Example

$L_2(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,, \|f\|_{L_2(\Omega)} < \infty \right\}$ with $\|f\|_{L_2(\Omega)} = \left(\int_\Omega f^2 \, \mathrm{d}\Omega\right)^{1/2}$

$L_p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,, \|f\|_{L_p(\Omega)} < \infty \right\}$ with $\|f\|_{L_p(\Omega)} = \left(\int_\Omega f^p \, \mathrm{d}\Omega\right)^{1/p}$

$W^{k,p}(\Omega) = \{ f : \Omega \to \mathbb{R} \,, D^j f \in L_p(\Omega), 0 \le j \le k \}$ with

$\|f\|_{W^{k,p}(\Omega)} = \left(\sum_{j=0}^{k} \|D^j f\|^p_{L_p(\Omega)} \right)^{1/p}$

Equivalence of norms

Two norms are equivalent when there exist two constants C_1, C_2 such that

$$C_1|||\boldsymbol{u}||| \leq \|\boldsymbol{u}\| \leq C_2|||\boldsymbol{u}|||$$

When two norms are equivalent, the measured function must be very small or very large in both norms.
In finite dimensional spaces (i.e., $\mathbb{R}^N$), all norms are equivalent

Example

Norms in $\mathbb{R}^N$

- $\|\boldsymbol{u}\|_\infty = \max_{1\leq k\leq N} |u_k|$
- $\|\boldsymbol{u}\|_1 = \sum_{k=1}^N |u_k|$
- $\|\boldsymbol{u}\|_2 = \left(\sum_{k=1}^N |u_k|^2\right)^{1/2}$
- $\|\boldsymbol{u}\|_p = \left(\sum_{k=1}^N |u_k|^p\right)^{1/p}$

Hilbert spaces

A Hilbert space is a complete inner product space. Important examples: $L_2(\Omega)$ is a Hilbert space with inner product

$$(f, g) = \int_\Omega f \cdot g \, \mathrm{d}\Omega$$

$H_1(\Omega) = W^{1,2}(\Omega)$ is a Hilbert space with inner product

$$(f, g) = \int_\Omega (f \cdot g + f' \cdot g') \, \mathrm{d}\Omega$$

$H_2(\Omega) = W^{2,2}(\Omega)$ is a Hilbert space with inner product

$$(f, g) = \int_\Omega (f \cdot g + f' \cdot g' + f'' \cdot g'') \, \mathrm{d}\Omega$$

- A function can be small in one norm and large in another norm (give example)
- In finite dimensions, all norms are equivalent. (Not in ∞ dimensions).
- The most important norm for this course is the H_1 norm.

Matrices

Definitions:

- A $n \times m$ matrix $\boldsymbol{M}$ is a linear map $\boldsymbol{M} : \mathbb{R}^m \to \mathbb{R}^n$.
- The kernel of $\boldsymbol{M}$ is the set $\mathrm{Ker}(\boldsymbol{M}) = \{\boldsymbol{v} \in \mathbb{R}^m : \boldsymbol{M}\boldsymbol{v} = \mathbf{0}_n\}$.
- The range of $\boldsymbol{M}$ is the set $\mathrm{Rg}(\boldsymbol{M}) = \{\boldsymbol{w} \in \mathbb{R}^n : \boldsymbol{w} = \boldsymbol{M}\boldsymbol{v} \text{ for some } \boldsymbol{v} \in \mathbb{R}^m\}$.
- A matrix is injective if $\boldsymbol{M}\boldsymbol{v} = \boldsymbol{M}\boldsymbol{w} \iff \boldsymbol{v} = \boldsymbol{w}$.
- A matrix is surjective if, for every $\boldsymbol{w} \in \mathbb{R}^m$, there exists $\boldsymbol{v} \in \mathbb{R}^n$ such that $\boldsymbol{M}\boldsymbol{v} = \boldsymbol{w}$.
- A matrix is bijective if it is both injective and surjective.

Matrix norms

Let $\|\cdot\|$ be a norm on the vector space $\mathcal{V}$. Then, a derived norm can be defined on any $\boldsymbol{M} : \mathcal{V} \to \mathcal{V}$ by the expression

$$\|\boldsymbol{M}\| = \max_{\boldsymbol{u} \in \mathcal{V}} \frac{\|\boldsymbol{M}\boldsymbol{u}\|}{\|\boldsymbol{u}\|}$$

It follows that

$$\|\boldsymbol{M}\boldsymbol{u}\| \leq \|\boldsymbol{M}\| \, \|\boldsymbol{u}\|$$

The condition number of matrix is $\kappa(\boldsymbol{M}) = \|M\| \|M^{-1}\|$.

Solvability in finite dimensions

Let M be a square matrix. The system $M\boldsymbol{v} = \boldsymbol{f}$ has a unique solution if any of the following holds

- M is bijective
- M is surjective
- M is injective
- $\det M \neq 0$

Assume that M is symmetric. Then the following condition implies that M is invertible

- M is definite positive

Stability

- A problem is stable when a small modification of the input data produces a small modification in its solution.
- Stability estimate 1: let $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$. If we perturb $\boldsymbol{f} \to \boldsymbol{f} + \delta\boldsymbol{f}$ and the new solution is $\boldsymbol{u} + \delta\boldsymbol{u}$, then

$$\frac{\|\delta\boldsymbol{u}\|}{\|\boldsymbol{u}\|} \leq \kappa(\boldsymbol{A})\frac{\|\delta\boldsymbol{f}\|}{\|f\|}$$

  hence $\kappa(\boldsymbol{A})$ is a measure of the stability of the system.
- Stability estimate 2: Let $\boldsymbol{A}$ be a square matrix with $\|\boldsymbol{A}\| \leq C$ and $\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{u} \geq \alpha\|\boldsymbol{u}\|^2$, then $\kappa(\boldsymbol{A}) \leq \frac{C}{\alpha}$.
- Stability estimate 3: let $\boldsymbol{A}$ be a square matrix with $\|\boldsymbol{A}\| \leq C$ and

$$\min_{\boldsymbol{u}} \max_{\boldsymbol{v}} \frac{\boldsymbol{v} \cdot \boldsymbol{A}\boldsymbol{u}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \geq \alpha > 0$$

  then $\kappa(\boldsymbol{A}) \leq \frac{C}{\alpha}$

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