MASTER IN SEISMIC ENGINEERING— E.T.S.I. INDUSTRIALES (U.P.M.)

DISCRETIZATION METHODS IN ENGINEERING

The FEM in 2 and 3D

Ignacio Romero
ignacio.romero@upm.es

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1. Review from vector/tensor calculus and algebra
2. The Poisson problem
3. The three formulations: strong, minimization, and weak
4. Computing the force vector and stiffness matrix

- You should have seen all the necessary calculus and algebra in the course by Prof. Montáns
- The fundamental identity we will need is the integration by parts formula in 2D or 3D: if Ω is a (simply connected open) set, and $u : \bar{\Omega} \to \mathbb{R}$ is a smooth function then, for all smooth $v : \bar{\Omega} \to \mathbb{R}$:

$$\int_{\Omega} v \, \Delta u \; \mathrm{d}\Omega = \int_{\partial\Omega} v(\nabla u \, \boldsymbol{n}) \, \mathrm{d}\mathit{\Gamma} - \int_{\Omega} \nabla v \cdot \nabla u \, \mathrm{d}\Omega$$

where $\boldsymbol{n}$ is the outward normal to $\partial\Omega$.

- It is one of the most common BVP of all mathematical physics.
- It represents diffusive problems (heat, concentration, membrane deformations, ...)
- It is the multidimensional generalization of the string problem we have studied this far.
- Strong form: let $\Omega \subset \mathbb{R}^d$, $(d = 2$ or $3)$ be a (simply connected open) set and $u : \bar{\Omega} \to \mathbb{R}$ which satisfies

$$
\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega \,, \\
u &= g \qquad \text{on } \partial_g \Omega \,, \\
\nabla u \, \boldsymbol{n} &= h \qquad \text{on } \partial_h \Omega \,,
\end{aligned}
$$

with $\partial_g\Omega \cap \partial_h\Omega = \emptyset$ and $\overline{\partial_g\Omega \cup \partial_h\Omega} = \partial\Omega$, and f, g, h given.

The Poisson problem - minimization problem

Let Ω be as before. Then, the solution to the Poisson problem is the function $u \in \mathcal{S}$ that minimizes the functional

$$\Pi[u] = \int_\Omega |\nabla u|^2 \, \mathsf{d}\Omega - \int_\Omega f \, u \, \mathsf{d}\Omega - \int_{\partial_h \Omega} h \, u \, \mathsf{d}\Omega \ ,$$

with the trial space $\mathcal{S}$ defined as

$$\mathcal{S} = \left\{ v : \bar{\Omega} \to \mathbb{R}, \ v = g \text{ on } \partial_g \Omega \right\}$$

In other words

$$u = \text{arg min}_{v \in \mathcal{S}} \Pi[v]$$

Let Ω and $\mathcal{S}$ be as before and define the test space

$$\mathcal{W} = \left\{ w : \bar{\Omega} \rightarrow \mathbb{R},\ w = 0 \text{ on } \partial_g \Omega \right\}$$

Then, the solution $u \in \mathcal{S}$ is the solution to the Poisson problem if for all $w \in \mathcal{W}$

$$\int_\Omega \nabla u \cdot \nabla w \, \mathsf{d}\Omega = \int_\Omega f \, w \, \mathsf{d}\Omega + \int_{\partial_h \Omega} h \, w \, \mathsf{d}\mathit{\Gamma}$$

- It is a multidimensional problem, similar to the Poisson problem, closely related with the Navier-Stokes equations.
- Strong form: let $\Omega \subset \mathbb{R}^d$, $(d = 2$ or $3)$ be a (simply connected open) set, $\boldsymbol{a} : \bar{\Omega} \rightarrow \mathbb{R}^3$ a given divergence-free velocity field, and $u : \bar{\Omega} \rightarrow \mathbb{R}$ which satisfies

$$
\begin{aligned}
-\kappa \, \Delta u + \boldsymbol{a} \cdot \nabla u &= f && \text{in } \Omega \, , \\
u &= g && \text{on } \partial\Omega \, ,
\end{aligned}
$$

with f, g given too.

Consider the functional spaces

$$\mathcal{S} = \left\{ u : \bar{\Omega} \to \mathbb{R}^3,\ u = g \text{ on } \partial\Omega \right\} \ , \quad \mathcal{W} = \left\{ w : \bar{\Omega} \to \mathbb{R}^3,\ w = 0 \text{ on } \partial\Omega \right\}$$

The weak solution to the advection-diffusion equation is the function $u \in \mathcal{S}$ that satisfies

$$\int_\Omega \left(\kappa \nabla u \cdot \nabla w - (\boldsymbol{a} \cdot \nabla w) u \right) \, \mathrm{d}\Omega = \int_\Omega f \, w \, \mathrm{d}\Omega$$

for all $w \in \mathcal{W}$.

- ▷ There is no minimization principle behind the advection-diffusion problem. Ritz method cannot be used to discretize it, but Galerkin's method can.

Minimization problem

$u = \arg\min_{w \in \mathcal{S}} \mathit{\Pi}(w)$

$\mathit{\Pi}(w) = \int_{\mathit{\Omega}} \kappa |\nabla w|^2 \, \mathrm{d}\mathit{\Omega} - \int_{\mathit{\Omega}} f \cdot w - \int_{\partial_h \mathit{\Omega}} h \cdot w \, \mathrm{d}\mathit{\Gamma}$

Ritz method

Finite element formulation

Galerkin method

Weak form: find u such that for all $w \in \mathcal{W}$

$\int_{\mathit{\Omega}} \kappa \nabla u \cdot \nabla w \, \mathrm{d}\mathit{\Omega} = \int_{\mathit{\Omega}} f \cdot w \, \mathrm{d}\mathit{\Omega} + \int_{\partial_h \mathit{\Omega}} h \cdot w \, \mathrm{d}\mathit{\Gamma}$

The Galerkin method seeks to find an approximation solution to the weak form of the Poisson problem by defining the finite dimensional spaces

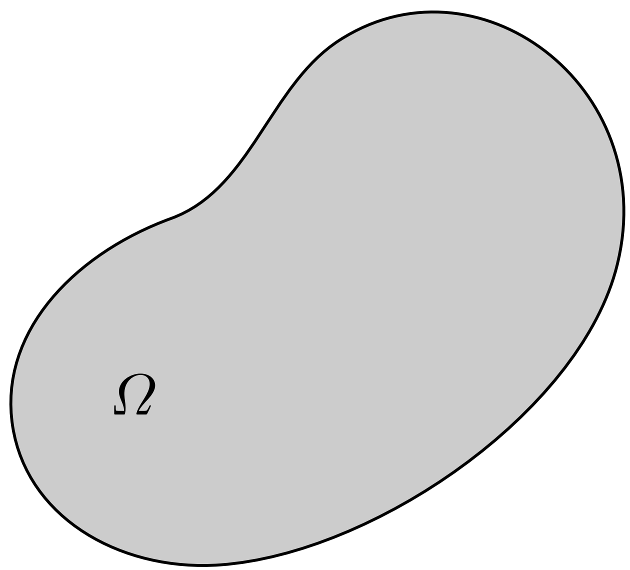
$$\mathcal{S}^h = \left\{ v^h \in \mathcal{S},\ v^h = \sum_{i=1}^{r} N_i(\boldsymbol{x}) v_i \right\} \subset \mathcal{S}$$

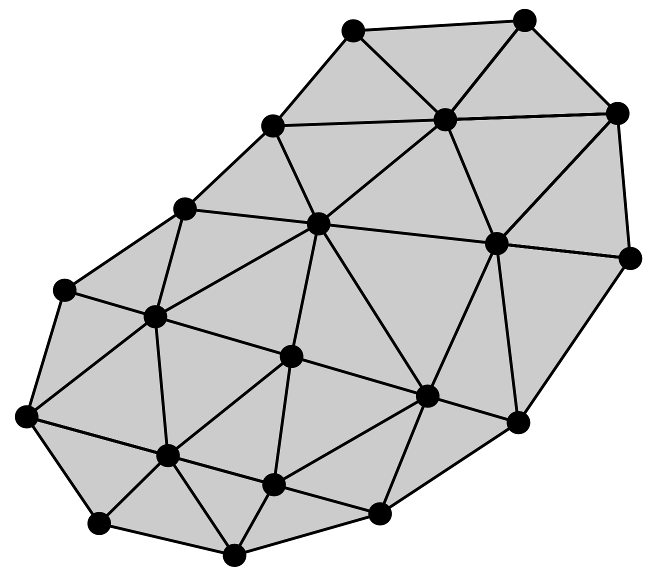
$$\mathcal{W}^h = \left\{ w^h \in \mathcal{W},\ w^h = \sum_{i=1}^{r} N_i(\boldsymbol{x}) w_i \right\} \subset \mathcal{W}$$

Then, the solution $u^h \in \mathcal{S}^h$ is the Galerkin solution to the Poisson problem if for all $w^h \in \mathcal{W}$

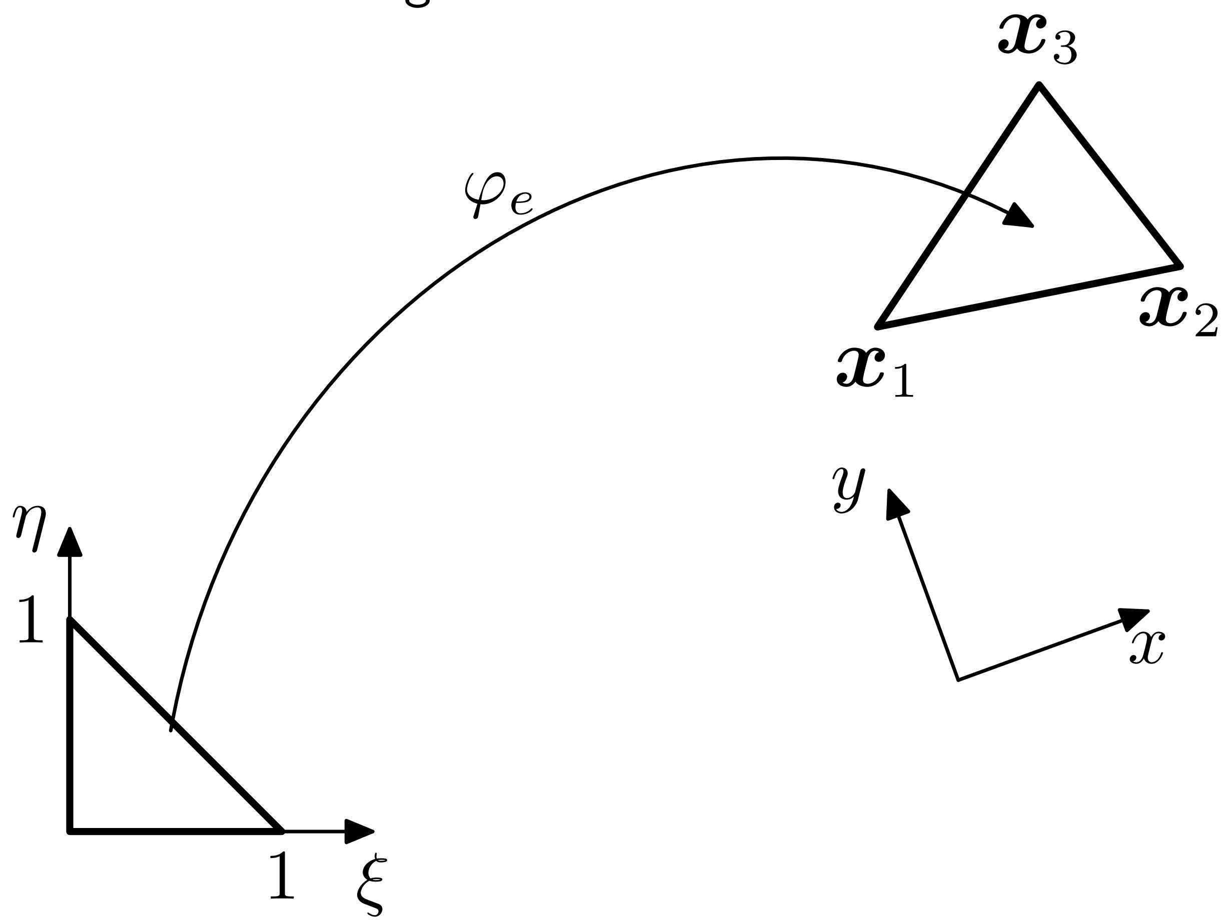
$$\int_\Omega \nabla u^h \cdot \nabla w^h \, \mathrm{d}\Omega = \int_\Omega f \, w^h \, \mathrm{d}\Omega + \int_{\partial_h \Omega} h \, w^h \, \mathrm{d}\Gamma$$

- The FEM can be understood as a particular instance of Galerkin method in which the interpolation functions $N_i : \bar{\Omega} \to \mathbb{R}$ are piecewise polynomials.
- To extend the definition of the shape functions we studied in 1D, we define a triangulation of the domain Ω.





As in 1D FE analysis, it is simpler to perform all the computations in a fixed element. For simplicity we consider the triangular element:



- These are linear functions, which are 1 in one of the nodes and 0 in the other two. In local coordinates they are simply:

$$\tilde{N}_e^1(\xi,\eta) = \xi$$
$$\tilde{N}_e^2(\xi,\eta) = \eta$$
$$\tilde{N}_e^3(\xi,\eta) = 1 - \xi - \eta$$

- The map $\boldsymbol{\varphi}_e : \triangle \to \Omega_e$ is defined by

$$\boldsymbol{\varphi}_e(\xi, \eta) = \sum_{a=1}^{3} \tilde{N}^a(\xi, \eta)\, \boldsymbol{x}_a$$

- The map $\boldsymbol{\varphi}_e$ uses for interpolation the same functions as the displacement field and the resulting finite element issaid to be *isoparametric*.
- The jacobian of this map is

$$\boldsymbol{J} = \left[\frac{\partial\,(x, y)}{\partial\,(\xi, \eta)} \right] = \begin{bmatrix} \frac{\partial\,\boldsymbol{\varphi}_e^x}{\partial\,\xi} & \frac{\partial\,\boldsymbol{\varphi}_e^x}{\partial\,\eta} \\ \frac{\partial\,\boldsymbol{\varphi}_e^y}{\partial\,\xi} & \frac{\partial\,\boldsymbol{\varphi}_e^y}{\partial\,\eta} \end{bmatrix}, \qquad \text{and we define } J = det\,\boldsymbol{J}$$

- For the computation of the stiffness matrix we need the gradient ∇N_e^i which can be computed as follows:

$$\nabla N_e^i = \left\{ \begin{array}{c} \frac{\partial N_e^i}{\partial x} \\ \frac{\partial N_e^i}{\partial y} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial \tilde{N}_e^i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \tilde{N}_e^i}{\partial \eta} \frac{\partial \eta}{\partial x} \\ \frac{\partial \tilde{N}_e^i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \tilde{N}_e^i}{\partial \eta} \frac{\partial \eta}{\partial y} \end{array} \right\} = \left[\begin{array}{cc} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial \tilde{N}_e^i}{\partial \xi} \\ \frac{\partial \tilde{N}_e^i}{\partial \eta} \end{array} \right\} = \boldsymbol{J}^{-T} \nabla_\xi \tilde{N}_e^i$$

$$\mathbf{f}_e = \int_{\Omega_e} f(x,y) \begin{Bmatrix} N_1^e(x,y) \\ N_2^e(x,y) \\ N^e 3(x,y) \end{Bmatrix} \mathsf{d}A$$

$$= \int_{\triangle} f(\boldsymbol{\varphi}_e(\xi,\eta)) \begin{Bmatrix} \tilde{N}_1^e(\xi,\eta) \\ \tilde{N}_2^e(\xi,\eta) \\ \tilde{N}_3^e(\xi,\eta) \end{Bmatrix} J(\xi,\eta)\, \mathsf{d}\triangle$$

$$\mathbf{k}_e = \int_{\Omega_e} \kappa \begin{Bmatrix} \nabla N_1^e(x,y) \\ \nabla N_2^e(x,y) \\ \nabla N_3^e(x,y) \end{Bmatrix} \begin{Bmatrix} \nabla N_1^e(x,y) \\ \nabla N_2^e(x,y) \\ \nabla N_3^e(x,y) \end{Bmatrix}^T \mathsf{d}A$$

$$= \int_{\triangle} \kappa \begin{Bmatrix} \nabla_\xi \tilde{N}_1^e(\xi,\eta) \\ \nabla_\xi \tilde{N}_2^e(\xi,\eta) \\ \nabla_\xi \tilde{N}_3^e(\xi,\eta) \end{Bmatrix} \boldsymbol{J}^{-1} \boldsymbol{J}^{-T} \begin{Bmatrix} \nabla_\xi \tilde{N}_1^e(\xi,\eta) \\ \nabla_\xi \tilde{N}_2^e(\xi,\eta) \\ \nabla_\xi \tilde{N}_3^e(\xi,\eta) \end{Bmatrix}^T J(\xi,\eta)\, \mathsf{d}\triangle$$

- Both for the calculation of the element contributions to the force vector and stiffness matrix we need to perform integrals of the form

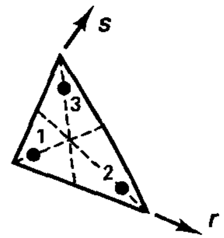
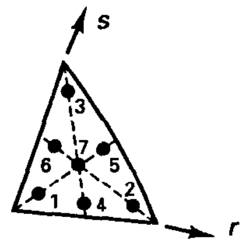
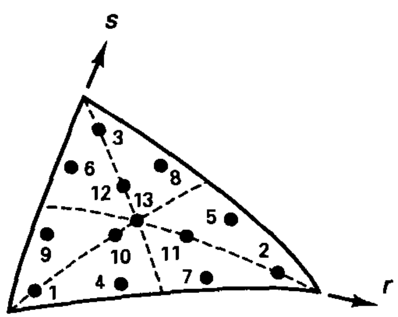
$$\int_{\triangle} q(\xi, \eta)\,\mathsf{d}\triangle$$

for some scalar, vector, or matrix function q defined on $\triangle$.

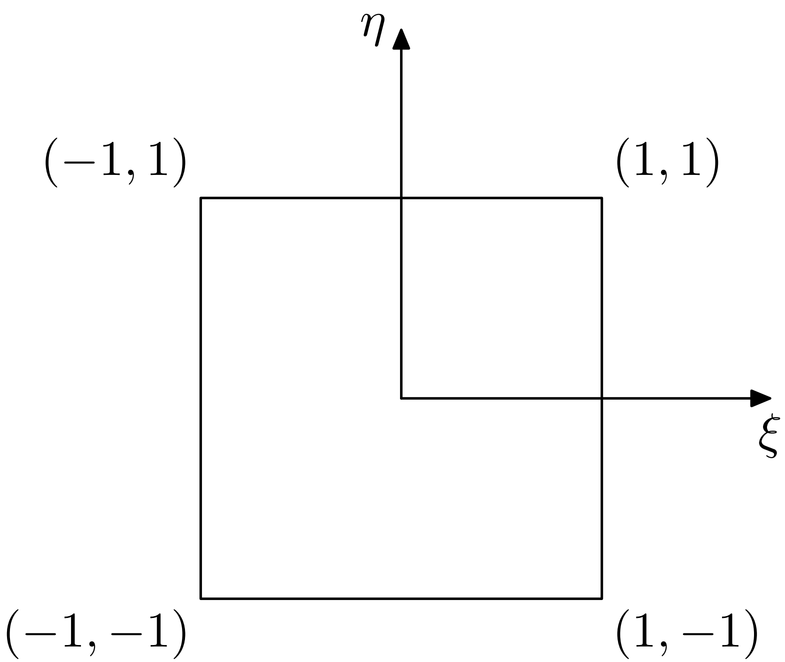
- In general, it is not possible to integrate this functions analytically so we must use numerical quadrature, as in the string, and approximate

$$\int_{\triangle} q(\xi, \eta)\,\mathsf{d}\triangle \approx \sum_{l=1}^{N} q(\xi_l, \eta_l) W_l$$

TABLE 5.8 *Gauss numerical integrations over triangular domains* $[\iint F\,dr\,ds = \frac{1}{2}\sum w_i F(r_i, s_i)]$

Integration order	Degree of precision	Integration points	r-coordinates	s-coordinates	Weights
3-point	2		$r_1 = 0.16666\ 66666\ 667$ $r_2 = 0.66666\ 66666\ 667$ $r_3 = r_1$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$	$w_1 = 0.33333\ 33333\ 333$ $w_2 = w_1$ $w_3 = w_1$
7-point	5		$r_1 = 0.10128\ 65073\ 235$ $r_2 = 0.79742\ 69853\ 531$ $r_3 = r_1$ $r_4 = 0.47014\ 20641\ 051$ $r_5 = r_4$ $r_6 = 0.05971\ 58717\ 898$ $r_7 = 0.33333\ 33333\ 333$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$ $s_4 = r_6$ $s_5 = r_4$ $s_6 = r_4$ $s_7 = r_7$	$w_1 = 0.12593\ 91805\ 448$ $w_2 = w_1$ $w_3 = w_1$ $w_4 = 0.13239\ 41527\ 885$ $w_5 = w_4$ $w_6 = w_4$ $w_7 = 0.225$
13-point	7		$r_1 = 0.06513\ 01029\ 022$ $r_2 = 0.86973\ 97941\ 956$ $r_3 = r_1$ $r_4 = 0.31286\ 54960\ 049$ $r_5 = 0.63844\ 41885\ 698$ $r_6 = 0.04869\ 03154\ 253$ $r_7 = r_5$ $r_8 = r_4$ $r_9 = r_6$ $r_{10} = 0.26034\ 59660\ 790$ $r_{11} = 0.47930\ 80678\ 419$ $r_{12} = r_{10}$ $r_{13} = 0.33333\ 33333\ 333$	$s_1 = r_1$ $s_2 = r_1$ $s_3 = r_2$ $s_4 = r_6$ $s_5 = r_4$ $s_6 = r_5$ $s_7 = r_6$ $s_6 = r_5$ $s_9 = r_4$ $s_{10} = r_{10}$ $s_{11} = r_{10}$ $s_{12} = r_{11}$ $s_{13} = r_{13}$	$w_1 = 0.05334\ 72356\ 088$ $w_2 = w_1$ $w_3 = w_1$ $w_4 = 0.07711\ 37608\ 903$ $w_5 = w_4$ $w_6 = w_4$ $w_7 = w_4$ $w_8 = w_4$ $w_9 = w_4$ $w_{10} = 0.17561\ 52574\ 332$ $w_{11} = w_{10}$ $w_{12} = w_{10}$ $w_{13} = -0.14957\ 00444\ 677$

- The four-node quadrilateral element is widely used, especially in elasticity.
- The reference (parent) domain is the biunit square



$$\tilde{N}^1(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 + \eta)$$
$$\tilde{N}^2(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$
$$\tilde{N}^3(\xi, \eta) = \frac{1}{4}(1 - \xi)(1 - \eta)$$
$$\tilde{N}^4(\xi, \eta) = \frac{1}{4}(1 + \xi)(1 - \eta)$$

- Are obtained simply by using Gauss quadrature in each direction (ξ, η).

- Elements of arbitrary order can be defined on the triangle and the quadrilateral.
- Quadratic triangles and quadrilateral are common but not so much elements of higher order.

- Strong and weak forms can be found for BVP in 2D and higher dimensions. Many problems also derive from a minimization principle, but not all.
- As in the 1D case, the FEM is just a particular instance of the Galerkin method for a specially simple case of piecewise linear interpolation functions.
- The structure of the force vector and stiffness matrix is the same as in the 1D case. Also now the element point of view simplifies the evaluation of integrals.
- Triangles and quads have standard interpolation functions and quadrature rules that can be used to standardize computations.
- Higher order extensions exist, but we don't study them.

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