Master in Seismic Engineering— E.T.S.I. Industriales (U.P.M.)

Discretization Methods in Engineering

An introduction to the mathematical theory of finite elements

Ignacio Romero
ignacio.romero@upm.es

October 8, 2014

Outline

1. Definitions
2. Unifying formulation for BVP and finite elements
3. Existence and uniqueness of BVP and finite elements
4. The optimality of the FE solution
5. What can go wrong

Let $\mathcal{V}$ be a vector space of (smooth) functions $\boldsymbol{u} : \Omega \to \mathbb{R}^k$, for Ω a domain in $\mathbb{R}^1, \mathbb{R}^2$, or $\mathbb{R}^3$ and $k = 1, 2,$ or 3.

- **Definition:** a function $L : \mathcal{V} \to \mathbb{R}$ is a linear form if

$$L(\alpha \boldsymbol{v} + \beta \boldsymbol{w}) = \alpha L(\boldsymbol{v}) + \beta L(\boldsymbol{w})$$

  for all $\alpha, \beta \in \mathbb{R}$ and $\boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.

- **Definition:** a function $B : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a bilinear form if

$$B(\alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w}) = \alpha B(\boldsymbol{u}, \boldsymbol{v}) + \beta B(\boldsymbol{u}, \boldsymbol{w}) \ ,$$
$$B(\boldsymbol{u}, \alpha \boldsymbol{v} + \beta \boldsymbol{w}) = \alpha B(\boldsymbol{u}, \boldsymbol{v}) + \beta B(\boldsymbol{u}, \boldsymbol{w}) \ ,$$

  for all $\alpha, \beta \in \mathbb{R}$ and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$.

- A bilinear form is said to be *symmetric* if

$$B(\boldsymbol{u}, \boldsymbol{v}) = B(\boldsymbol{v}, \boldsymbol{u})$$

  for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$.

- *All* the boundary value problems seen in this course are of the form: find $\boldsymbol{u} \in \mathcal{V}$ such that, for all $\boldsymbol{v} \in \mathcal{V}$,

$$\boxed{B(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v})}$$

  where L is a (continuous) linear form and B is a bilinear form on $\mathcal{V}$.

Hence, if we are able to say something about this problem, it is valid for all the problems studies this far.

- All the bilinear forms studied, except for that of the advection diffusion problem, are symmetric.

If we are able to say even more things about symmetric problems, these conclusions will be valid in all the corresponding cases.

(Note: in fact, we considered slightly more complex formulations in which the test and weighting spaces were different, due to the boundary conditions, but it is essentially the same as above).

- *All* the finite element formulations seen in this course are of the form: define a finite dimensional subspace $\mathcal{V}^h \subset \mathcal{V}$ and find $\boldsymbol{u}^h \in \mathcal{V}^h$ such that for all $\boldsymbol{v}^h \in \mathcal{V}^h$

$$B(\boldsymbol{u}^h, \boldsymbol{v}^h) = L(\boldsymbol{v}^h)$$

- The linear and bilinear forms are identical to that of the continuous problem, only the solution and weighting space changes
- Moreover, all the finite element solutions computed can be written in matrix form

$$\mathsf{KU} = \mathsf{F}$$

with $\mathsf{K}^{ab} = B(N^a, N^b)$, $\mathsf{F}^a = L(N^a)$.

- **Lax-Milgram theorem:** Let $B : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ be a bilinear form satisfying

$$|B(\boldsymbol{u}, \boldsymbol{v})| \leq C|||\boldsymbol{u}|||\ |||\boldsymbol{v}||| \qquad (*)$$
$$B(\boldsymbol{u}, \boldsymbol{u}) \geq \alpha\, |||\boldsymbol{u}|||^2 \qquad (**)$$

  for all $\boldsymbol{u} \in \mathcal{V}$, with $\alpha > 0$ being a constant, and $|||\cdot|||$ some natural norm of the problem, then the variational problem of finding $\boldsymbol{u} \in \mathcal{V}$ satisfying $B(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v})$ for all $\boldsymbol{v} \in \mathcal{V}$ has a unique solution which, furthermore, depends continuously on L (whether B is symmetric or not).

- Remarks:
  - A bilinear form that satisfies $(*)$ is said to be *continuous*.
  - A bilinear form that satisfies $(**)$ is said to be *elliptic*.
  - There are additional technical conditions on B and the space $\mathcal{V}$.
  - The bilinear form need not be symmetric

- The finite element problem has a unique solution if the matrix **K** is invertible.
- If the B form is elliptic, for every nonzero $\boldsymbol{u}^h \in \mathcal{V}$ and its corresponding vector of nodal degrees of freedom **U**, we have

$$\mathbf{U}^T\mathbf{K}\mathbf{U} = B(\boldsymbol{u}^h, \boldsymbol{u}^h) \geq \alpha \left\|\boldsymbol{u}^h\right\|^2 > 0$$

  concluding that **K** is positive definite, and hence invertible.

▷ Even if the FE has a unique solution, it might not be good enough. The solution needs to be well-conditioned.

- If the bilinear form B is symmetric, then there is a functional Π whose minimizer is the solution to the variational problem.

$$\boldsymbol{u} = \underset{\boldsymbol{v} \in \mathcal{V}}{\arg\min}\ \Pi[\boldsymbol{v}]$$

- The form of the functional is

$$\Pi[\boldsymbol{u}] = \tfrac{1}{2} B(\boldsymbol{u}, \boldsymbol{u}) - L(\boldsymbol{u})$$

- If the bilinear form is not symmetric, then there is not a corresponding minimization principle.

- For minimization problems, it is possible to study the existence of solutions directly from the functional Π, although it is more complicated.
- The advantage is that there exist results *even for nonlinear problems*.
- One needs to show that Π is regular, coercive, and lower semicontinuous (Explain).

The finite element method always gives approximate solutions which are, in some sense, optimal. Let $\mathcal{V}^h \subset \mathcal{V}$ be a finite element space.

- ▷ **Optimality interpretation 1:** If the problem can be stated as a minimization problem, then the finite element solution is, by definition, the one of smallest energy among all functions in $\mathcal{V}^h$, hence optimal in some sense.

It can be shown that if the bilinear form is elliptic and symmetric it defines an inner product and a norm in $\mathcal{V}$, the so called "energy inner product" and "energy norm", defined respectively as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_E = B(\boldsymbol{u}, \boldsymbol{v}) \ , \qquad \|\boldsymbol{u}\|_E = \sqrt{B(\boldsymbol{u}, \boldsymbol{u})}$$

- **Optimality interpretation 2:** Let $e = \boldsymbol{u}^h - \boldsymbol{u}$ be the error in the finite element solution. It is energy-orthogonal to $\mathcal{V}^h$.

$$\langle e, \boldsymbol{v}^h \rangle_E = 0 \text{ for all } \boldsymbol{v}^h \in \mathcal{V}^h$$

Proof: Take $B(\boldsymbol{u}, \boldsymbol{v}) = L(\boldsymbol{v})$ and choose $\boldsymbol{v} = \boldsymbol{v}^h$. Substract $B(\boldsymbol{u}^h, \boldsymbol{v}^h) = L(\boldsymbol{v}^h)$ and use the definition of bilinearity.

- **Optimality interpretation 3:** The error made by the finite element solution is the smallest (measured in the energy norm) among all the errors made by other functions in $\mathcal{V}^h$.

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_E \leq \|\boldsymbol{u} - \boldsymbol{w}^h\|_E \ , \qquad \text{for all } \boldsymbol{w}^h \in \mathcal{V}^h$$

Proof:

$$\|e\|_E^2 = B(e, e) = B(e, \boldsymbol{u}) = B(e, \boldsymbol{u} - \boldsymbol{w}^h) \leq \|e\|_E \, \|\boldsymbol{u} - \boldsymbol{w}^h\|_E$$

It can be shown that a sufficiently smooth function $\boldsymbol{u}$ can be approximated by a piecewise polynomial $\Pi^h \boldsymbol{u}$ with an error of the form

$$\|\boldsymbol{u} - \Pi^h \boldsymbol{u}\|_E \leq C h^k |||\boldsymbol{u}|||$$

where C is a constant, h is a measure of the size of the elements, $k \geq 1$ is a constant that depends on the order of the polynomials in the FE space, and $|||\cdot|||$ is a (Sobolev) norm. Combining this *interpolation estimate* with the last optimality condition we obtain

- **Convergence theorem:** The error in the finite element solution satisfies

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_E \leq C\, h^k |||\boldsymbol{u}|||$$

This proves *convergence* of the finite element method, that is, when $h \to 0$, the error goes to zero too.

What can go wrong (1)

- The bilinear form is not elliptic.
  - ▷ *Comment:* Ellipticity is a sufficient condition for solvability and uniqueness of the variational problem, but it is not necessary. A weaker condition exists but it is more complicated to analyze.
  - ▷ *Caution:* Some finite elements (so called "mixed methods") fall within this category. Their use is tricky and their analysis complex. Beware!.

What can go wrong (2)

- The bilinear form is elliptic, but not symmetric and does not define an energy norm.
  - ▷ *Comment:* The analysis is similar to what has been presented, but the error estimates have to be done in another norm, natural to the problem, which obviously is not defined by $B(\cdot, \cdot)$. Sobolev norms are the natural ones.
  - ▷ *Caution:* There might be issues with the uniformity of the bounds.

- The form is elliptic, it is symmetric, so the finite element solution is the best in its solution space. What if the finite element space is inappropriate for the problem?
  - ▷ *Comment:* The solution is still the best in some sense, but it might be useless.
  - ▷ *Caution:* This is the reason for the phenomenon known as "locking": the finite element solution is extremely stiff. It can be solved by changing the bilinear form to something which is equivalent in the continuous formulation but different when discretized.

- Timoshenko beam:

$$\Pi(u,\theta) = \int_0^L \left(\frac{EI}{2}(\theta')^2 + \frac{GA_s}{2}(u'-\theta)^2 \right) \mathsf{d}x - \int_0^L qu \, \mathsf{d}x - \bar{M}\theta(L) - \bar{F}u(L)$$
$$= \frac{EI}{2} \int_0^L \left((\theta')^2 + \frac{C}{t^2}(u'-\theta)^2 \right) \mathsf{d}x - \int_0^L qu \, \mathsf{d}x - \bar{M}\theta(L) - \bar{F}u(L)$$

- Bilinear form:

$$B(u,\theta;v,\alpha) = \int_0^L \left(EI \; \theta'\alpha' + GA_s \; (u'-\theta)(v'-\alpha) \right) \mathsf{d}x$$

- Solution in the thin limit $t/L \to 0$ requires $u' = \theta$ and u is the minimizer of

$$\Pi_0(\theta) = \int_0^L \frac{EI}{2}(\theta')^2 \, \mathsf{d}x - \int_0^L qu \, \mathsf{d}x - \bar{M}\theta(L) - \bar{F}u(L)$$

with $u' = \theta$

- In general, to enforce a constraint on the solution, we can use Lagrange multipliers

$$\Pi_3(\theta, u, q) = \int_0^L \left(\frac{EI}{2} (\theta')^2 + q(u' - \theta) \right) \mathrm{d}x - \int_0^L q u \, \mathrm{d}x - \bar{M}\theta(L) - \bar{F}u(L)$$

with

$$(\theta, u, q) = \inf_{\theta, u} \sup_{q} \Pi_3(\theta, u, q)$$

- The problem is no longer one of minimization, but a saddle point.
- The theory of saddle points is more subtle.
- The weak form is of the form:

$$\begin{aligned} a(u, \theta; v, \alpha) + b(u, \theta; p) &= f(v, \alpha) \\ b(u, \theta; q) &= g(q) \end{aligned}$$

- The bilinear form $a(\cdot, \cdot)$ must verify a coercivity condition and $b(,)$ an inf-sup condition, known as the LBB condition.

Master in Seismic Engineering— E.T.S.I. Industriales (U.P.M.)

Discretization Methods in Engineering

An introduction to the mathematical theory of finite elements

Ignacio Romero

ignacio.romero@upm.es

October 8, 2014