Master in Seismic Engineering— E.T.S.I. Industriales (U.P.M.)

Discretization Methods in Engineering

Finite elements for elasticity

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Recall from the elementary theory of elasticity: Given a body occupying a domain Ω in $\mathbb{R}^2$ or $\mathbb{R}^3$ with points $\boldsymbol{x} \in \Omega$ (we focus on $\mathbb{R}^3$).

- The boundary $\partial\Omega$ can be split as in

$$\partial\Omega = \overline{\partial_g\Omega \cup \partial_h\Omega}\ , \quad \text{with } \partial_g\Omega \cap \partial_h\Omega = \emptyset\ ,$$

- The displacement vector field is denoted $\boldsymbol{u} : \Omega \to \mathbb{R}^3$.
- The stress tensor is denoted $\boldsymbol{\sigma}$.
- The strain tensor is $\boldsymbol{\varepsilon}$.
- The stress-strain relation is given a constitutive law of the form

$$\boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon},$$

with $\mathbb{C}$ being the (4th order) tensor of elasticities

$$\mathbb{C} = \lambda\, \mathbf{1} \otimes \mathbf{1} + 2\mu\, \mathbb{I}$$

and λ, μ are the Lamé constants (the tensor of elasticities has major and minor symmetries).

Governing equations (strong form)

Let $\boldsymbol{u} : \Omega \rightarrow \mathbb{R}^3$ be a displacement field on the deformable body Ω which has a known value $\boldsymbol{g}$ on $\partial_g \Omega$. The body is subjected to body forces $\boldsymbol{f}$ and tractions $\boldsymbol{h}$ on $\partial_u \Omega$. The vector field $\boldsymbol{u}$ is the solution to the elastic problem if it satisfies:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} &= \mathbf{0} && \text{in } \Omega \\
\boldsymbol{\sigma} &= \mathbb{C} : \boldsymbol{\varepsilon} \\
\boldsymbol{\varepsilon} &= \nabla^S \boldsymbol{u} \\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \partial_g \Omega \\
\boldsymbol{\sigma} \boldsymbol{n} &= \boldsymbol{h} && \text{on } \partial_h \Omega
\end{aligned}
$$

Define the functional space

$$\mathcal{S} = \left\{ \boldsymbol{w} : \Omega \to \mathbb{R}^3,\ \boldsymbol{w} = \boldsymbol{g} \text{ on } \partial_g \Omega \right\}$$

The vector field $\boldsymbol{u} : \Omega \to \mathbb{R}^3 \in \mathcal{S}$ is the solution to the elastic problem if it minimizes the potential energy

$$\Pi[\boldsymbol{u}] = \int_\Omega \tfrac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \, \mathrm{d}V - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u} \, \mathrm{d}V - \int_{\partial_h \Omega} \boldsymbol{h} \cdot \boldsymbol{u} \, \mathrm{d}A$$

among all the functions in the space $\mathcal{S}$, with $\boldsymbol{\varepsilon} = \nabla^S \boldsymbol{u}$.

Define the weighting functional space

$$\mathcal{W} = \left\{ \boldsymbol{w} : \Omega \to \mathbb{R}^3,\ \boldsymbol{w} = \mathbf{0} \text{ on } \partial_g \Omega \right\}$$

Then, $\boldsymbol{u} \in \mathcal{S}$ is the solution to the elastic problem if it satisfies the variational equation

$$\int_\Omega \nabla^S \boldsymbol{u} : \mathbb{C} : \nabla^S \boldsymbol{w} \,\mathrm{d}V = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w} \,\mathrm{d}V + \int_{\partial_h \Omega} \boldsymbol{h} \cdot \boldsymbol{w} \,\mathrm{d}A$$

for all $\boldsymbol{w} \in \mathcal{W}$.

Define the finite dimensional spaces

$$\mathcal{S}^h = \left\{ \boldsymbol{u}^h \in \mathcal{S},\ \boldsymbol{u}^h(\boldsymbol{x}) = \sum_{a=1}^{r} N_a(\boldsymbol{x})\, \boldsymbol{u}_a \right\} \ ,$$

$$\mathcal{W}^h = \left\{ \boldsymbol{w}^h \in \mathcal{W},\ \boldsymbol{w}^h(\boldsymbol{x}) = \sum_{a=1}^{r} N_a(\boldsymbol{x})\, \boldsymbol{w}_a \right\} \ ,$$

where the functions $N_a : \Omega \to \mathbb{R}$ are the (piecewise linear) usual shape functions.

Then, the finite element solution $\boldsymbol{u}^h \in \mathcal{S}^h$ to the elastic problem is the one that satisfies

The finite element formulation is

$$\int_\Omega \nabla^S \boldsymbol{u}^h : \mathbb{C} : \nabla^S \boldsymbol{w}^h \, \mathrm{d}V = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{w}^h \, \mathrm{d}V + \int_{\partial_h \Omega} \boldsymbol{h} \cdot \boldsymbol{w}^h \, \mathrm{d}A$$

for all $\boldsymbol{w}^h \in \mathcal{W}^h$.

Replacing the expression for the finite element functions, and using the properties of the elasticity tensor we obtain

$$\sum_{a,b=1}^{r} \int_\Omega (\boldsymbol{w}_a \otimes \nabla N_a) : \mathbb{C} : (\boldsymbol{u}_b \otimes \nabla N_b)\, \mathsf{d}V = \sum_{a=1}^{r} \boldsymbol{w}_a \cdot \left\{ \int_\Omega \boldsymbol{f} N_a \, \mathsf{d}V + \int_{\partial_h \Omega} \boldsymbol{h} N_a \, \mathsf{d}A \right\}$$

which can be written as

$$\sum_{a,b=1}^{r} \boldsymbol{w}_a \cdot \int_\Omega \boldsymbol{T}_{ab} \, \mathsf{d}V \cdot \boldsymbol{u}_b = \sum_{a=1}^{r} \boldsymbol{w}_a \cdot \left\{ \int_\Omega \boldsymbol{f} N_a \, \mathsf{d}V + \int_{\partial_h \Omega} \boldsymbol{h} N_a \, \mathsf{d}A \right\}$$

with

$$\boldsymbol{T}_{ab} = \lambda \nabla N_a \otimes \nabla N_b + \mu \nabla N_b \otimes \nabla N_a + \mu \, (\nabla N_a \cdot \nabla N_b) \mathbf{1} \ .$$

The finite element local force vector and stiffness matrix are, respectively,

$$\boldsymbol{f}_e = \int_{\Omega_e} \mathbf{N}^e \boldsymbol{f} \, \mathrm{d}V \ ,$$

$$\boldsymbol{k}_e = \int_{\Omega_e} \begin{bmatrix} \boldsymbol{T}_e^{11} & \boldsymbol{T}_e^{12} & \dots & \boldsymbol{T}_e^{1n_{node}} \\ \boldsymbol{T}_e^{21} & \boldsymbol{T}_e^{22} & \dots & \boldsymbol{T}_e^{2n_{node}} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{T}_e^{n_{node}1} & \boldsymbol{T}_e^{n_{node}2} & \dots & \boldsymbol{T}_e^{n_{node}n_{node}} \end{bmatrix} \mathrm{d}V$$

with

$$\boldsymbol{N}^e = \begin{bmatrix} N_1 \mathbf{1} \\ N_2 \mathbf{1} \\ \vdots \\ N_{n_{node}} \mathbf{1} \end{bmatrix}, \qquad \boldsymbol{T}_e^{ij} = \lambda \nabla N_i \otimes \nabla N_j + \mu \nabla N_j \otimes \nabla N_i + \mu \left(\nabla N_i \cdot \nabla N_j \right) \mathbf{1} \ .$$

- ▷ The vector $\boldsymbol{f}_e$ has length $n_{node} \times 3$.
- ▷ The matrix $\boldsymbol{k}_e$ has dimensions $(n_{node} \times 3) \times (n_{node} \times 3)$.

Define Voigt notation

$$\boldsymbol{\varepsilon} \to \{\boldsymbol{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} \qquad \boldsymbol{\sigma} \to \{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 2\sigma_{xy} \end{Bmatrix} \qquad \{\boldsymbol{\sigma}\} = [\mathbf{C}]\,\{\boldsymbol{\varepsilon}\}$$

Strain of a finite element function $\{\boldsymbol{\varepsilon}[\boldsymbol{u}^h]\}$:

$$\begin{Bmatrix} u^h_{x,x} \\ u^h_{y,y} \\ u^h_{x,y} + u^h_{y,x} \end{Bmatrix} = \begin{Bmatrix} \sum_a N^a_{,x} u^a_x \\ \sum_a N^a_{,y} u^a_y \\ \sum_a (N^a_{,y} u^a_x + N^a_{,x} u^a_y) \end{Bmatrix} = \sum_a \begin{bmatrix} N^a_{,x} & 0 \\ 0 & N^a_{,y} \\ N^a_{,y} & N^a_{,x} \end{bmatrix} \begin{Bmatrix} u^a_x \\ u^a_y \end{Bmatrix} = \sum_a \mathbb{B}_a \boldsymbol{u}_a$$

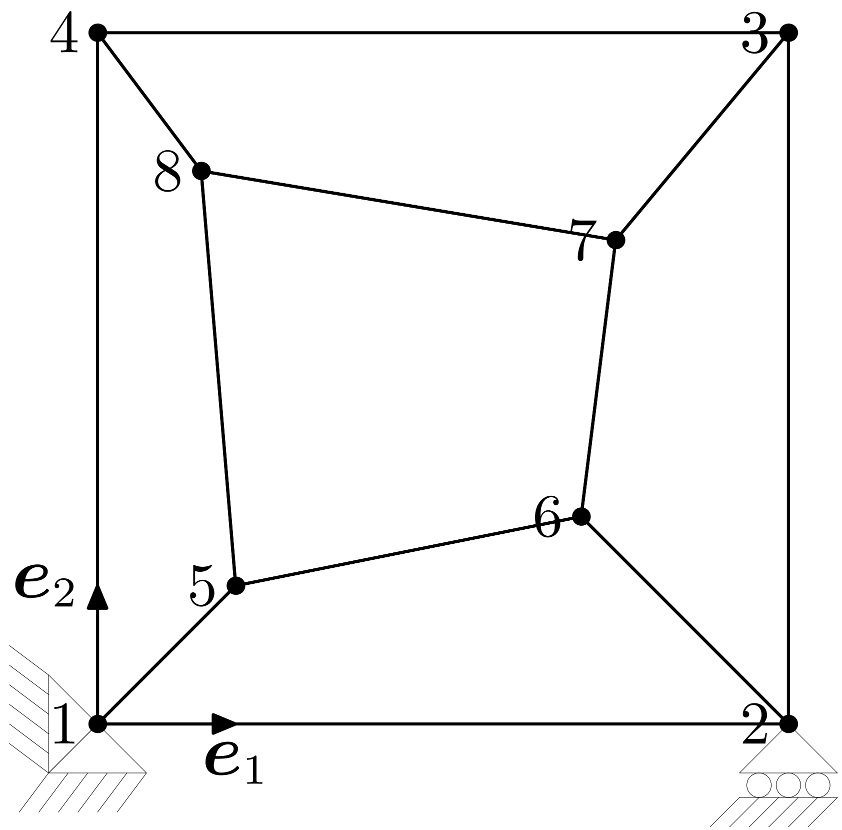
Virtual work:

$$\int_{\Omega_e} \nabla^s \boldsymbol{w}^h : \mathbb{C} : \nabla^s \boldsymbol{u}^h \, \mathrm{d}\Omega = \int_{\Omega_e} \{\boldsymbol{\varepsilon}[\boldsymbol{w}^h]\}^T \{\boldsymbol{\sigma}\} \, \mathrm{d}\Omega = \sum_{ab} \int_{\Omega_e} (\mathbb{B}_a \boldsymbol{w}_a)^T \mathbf{C} (\mathbb{B}_b \boldsymbol{u}_b) \, \mathrm{d}\Omega$$

$$= \sum_{ab} \boldsymbol{w}_a^T \int_{\Omega_e} \mathbb{B}_a^T \mathbf{C} \mathbb{B}_b \, \mathrm{d}\Omega \boldsymbol{u}_b = \sum_{ab} \boldsymbol{w}_a^T \boldsymbol{K}_{ab} \boldsymbol{u}_b = \sum_{ab} \boldsymbol{w}_a^T \boldsymbol{T}_{ab} \boldsymbol{u}_b$$

- Recall that, since the finite element solution $\boldsymbol{u}^h$ is just a Ritz solution, it is the best possible solution (in the energy sense) among all functions in $\mathcal{S}^h$.
- If the exact solution to an elastic problem $\boldsymbol{u}$ happens to fall within $\mathcal{S}^h$ then, $\boldsymbol{u}^h$ must be identical to $\boldsymbol{u}$ and thus the FE must give the exact solution.
- Consider a solid whose boundary conditions are consistent with a linear field of displacement. Then the unique exact solution $\boldsymbol{u}$ must be in the finite element space $\mathcal{S}^h$ and thus $\boldsymbol{u}^h = \boldsymbol{u}$.

- Consider a patch of elements (possibly distorted) like the one of the figure without any rigid body modes



For an mesh of elements with constant elasticities:

- Impose, *on the boundary nodes*, any function of the form

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{Bmatrix} a_o + a_1 x + a_2 y \\ b_o + b_1 x + b_2 y \end{Bmatrix}$$

  The displacement on the inner nodes must be exact and the stresses must be constant
- There are six possible “simple” tests which can be done independently.
- The patch test has been employed as a means of ascertaining the correctness of finite element formulations.