Locking

Ignacio Romero

Technical University of Madrid

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Do finite elements always work?

- For minimization problems the FEM it always works. For other Galerkin approximations, it need not work.
- Meaning of “works”: The finite element method, like any other Ritz method, yields the optimal approximation among those functions in the trial, and “optimality” refers to the energy norm.

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- Meaning of "works": The finite element method, like any other Ritz method, yields the optimal approximation among those functions in the trial, and "optimality" refers to the energy norm.
- What if the energy norm is "bad"? (meaning that it is not similar to the H^1-norm.
    - ▷ Then the "best" solution will look terrible.
    - ▷ This is a classical issue in finite elements called locking and one that can not be explained in 1D problems.

Geometric interpretation

$$C_1\|u^h\|_E \leq \|u^h\|_{H^1} \leq C_2\|u^h\|_E$$

Energetic interpretation

In 2D and 3D elasticity the energy density can be spit as

$$W(\boldsymbol{\varepsilon}) = W^{dev}(\boldsymbol{e}) + W^{vol}(\theta)$$

with $\theta = \text{tr}(\boldsymbol{\varepsilon})/3$ and $\boldsymbol{e} = \boldsymbol{\varepsilon} - \theta \boldsymbol{I}$. But

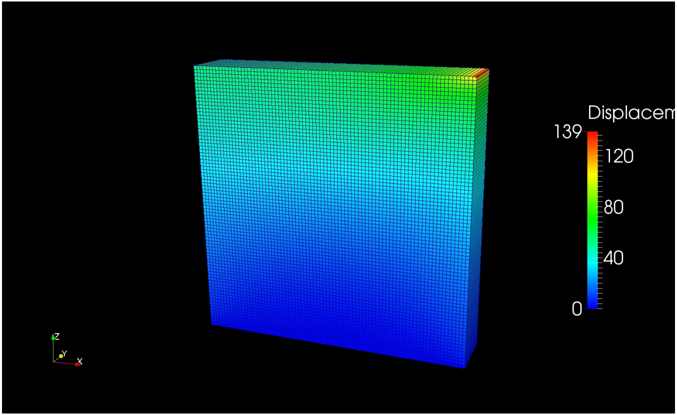
$$W^{dev}(\boldsymbol{e}) = \mu \|\boldsymbol{e}\|^2 \,, \qquad W^{vol}(\theta) = \frac{\kappa}{2}\theta^2$$

When $\nu \to 0.5$, the ratio $\kappa/\mu \to \infty$. Then, if the energy functional is to be minimized:

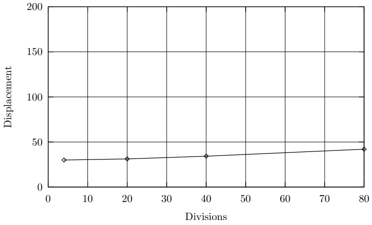
$$\Pi(\boldsymbol{u}^h) = \int_\Omega (W^{dev}(\boldsymbol{e^h}) + W^{vol}(\theta^h))\, \mathrm{d}\Omega + \Pi_{ext}(\boldsymbol{u}^h)$$

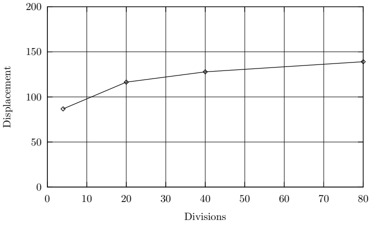
which term has the most relevant contribution? What happens with triangles/tetrahedra? (sketch)

Example with $\nu = 0.4999$



Results: Standard vs. mixed





Is this important

- Locking occurs for materials with $\mu/\kappa \to 0$, but also with beams with $t/L \to 0$, or plates, and other problems with a parameter dependency.
- In solids we know that during plastic and viscoelastic flow $\mu_{app}/\kappa \to 0$. In practice, this is a very important and common situation, even more because it is "hidden".
- For quite a while, locking puzzled finite element developers. Until it was well understood many solutions were tried. Some worked (miraculously!) and some not quite.
- All commercial codes include element formulation that do not lock.

Solution 1: selective reduced integration

- Recall the numerical evaluation of the energy:

$$\Pi(\boldsymbol{u}^h) \approx \sum_{l=1}^{N_d} W^{dev}(\boldsymbol{e}^{\boldsymbol{h}}(\boldsymbol{x}_l)) J_l W_l + \sum_{l=1}^{N_v} W^{vol}(\theta^h(\boldsymbol{x}_l)) J_l W_l + \Pi_{ext}(\boldsymbol{u}^h)$$

- Almost by luck, people noticed that using a low-accuracy quadrature rule for the volumetric term (for quads) solved the problem. Without knowing why. E.g.: for 2D, $N_d = 4, N_v = 1$.
- To be consistent, the forces and stiffness terms must be modified accordingly.
- If a low accuracy quadrature is used for all terms, the resulting elements are called "reduced integration elements". Sometimes they "work", sometimes they don't.
- If a low accuracy quadrature is used for the volumetric terms only, the resulting elements are called "selective reduced integration elements". They always "work" for quads.
- "Work" means that they do not lock.

Solution 2: mixed methods

- If the Ritz method give a solution which is not good, despite being the best, one can not hope to improve it ... unless we don't use Ritz method.
- The idea is to go back to the continuum formulation and find an equivalent formulation that describes the same problem ... whose discretization is not equivalent to the standard FE one.
- For example, for elasticity: Find $\boldsymbol{u} \in H^1_o(\Omega)$ and $p \in L^2(\Omega)$ that make the following functional stationary:

$$\Pi(\boldsymbol{u}, p) = \int_\Omega \left(W^{dev}(\boldsymbol{e}(\boldsymbol{u})) + p\,\theta(\boldsymbol{u}) - \chi(p) \right) \mathrm{d}V + \Pi_{ext}(\boldsymbol{u})$$

- This is a mixed variational principle, because of the different nature of the fields $\boldsymbol{u}$ and p.
- The function χ is the complementary energy $\chi(p) = \dfrac{1}{2\kappa} p^2$.
- The solution no longer gives the energy minimizer!

FE for mixed variational principles

- One can formulate a Galerkin method approximating both solution spaces

$$\boldsymbol{u}^h \in \mathcal{V}^h \subset H_0^1(\Omega) , \qquad p^h \in \mathcal{P}^h \subset L^2(\Omega)$$

- Then, find $\boldsymbol{u}^h$ and p^h that make $\Pi(\boldsymbol{u}^h, p^h)$ stationary. These conditions are always of the form:

$$\begin{aligned} a(\boldsymbol{u}^h, \boldsymbol{w}^h) + b(\boldsymbol{w}^h, p^h) &= f(\boldsymbol{w}^h) \\ b(\boldsymbol{u}^h, q^h) &= g(q^h) \end{aligned}$$

- The issue is that $\mathcal{V}^h$ and $\mathcal{P}^h$ can not be independently chosen. Even if $a(\cdot,\cdot)$ is elliptic, these spaces must satisfy the inf-sup condition:

$$\inf_{p \in \mathcal{P}^h} \sup_{\boldsymbol{u}^h \in \mathcal{V}^h} \frac{b(\boldsymbol{u}^h, p^h)}{\|\boldsymbol{u}^h\|_{H_o^1(\Omega)} \|p^h\|_{L^2(\Omega)}} \geq \beta > 0$$

for some β independent of h

Mixed methods

Some stable $\mathcal{V}^h, \mathcal{P}^h$ pairs for quadrilaterals/hexahedra are

$\mathcal{V}^h$	$\mathcal{P}^h$
Q^2	Q^1
Q^2	P^1 (discontinuous)

Some stable $\mathcal{V}^h, \mathcal{P}^h$ pairs for triangle/tets are

$\mathcal{V}^h$	$\mathcal{P}^h$
P^2	P^0 (discontinuous)
P^{2+}	P^1 (discontinuous)
P^{2+}	P^1

Other formulations

There has been an enormous effort to develop robust finite elements. Other solutions:

- Incompatible modes element.
- Enhanced strain elements.
- Stabilized formulations.
- Residual free bubbles.
- Hybrid methods.

Most commercial software have implemented one (or more) of these.

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